พฤติกรรมแบบเกาส์ของเคอร์เนลความร้อน
Gaussian Behavior of Heat Kernels

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บทคัดย่อ

สมการความร้อนคือสมการที่ใช้อธิบายการกระจายความร้อนบนปริภูมิหนึ่งๆ ณ เวลาใดๆ โดยทั่วไปคำตอบของสมการความร้อนจะอยู่ในรูปปริพันธ์ของผลคูณระหว่างฟังก์ชันความร้อนเริ่มต้นและเคอร์เนลความร้อน ดังนั้นพฤติกรรมของเคอร์เนลความร้อนจึงมีผลกระทบโดยตรงต่อพฤติกรรมของคำตอบของสมการความร้อน บทความนี้จะยกตัวอย่างความสัมพันธ์ข้างต้น ซึ่งก็คือความสัมพันธ์ระหว่างพฤติกรรมแบบเกาส์ของเคอร์เนลความร้อนและอสมการฮาร์แนค รวมถึงผลกระทบที่พฤติกรรมทั้งสองมีต่อสมบัติทางเรขาคณิต ได้แก่สมบัติทวีคูณและอสมการปวงกาเรของปริภูมินั้นด้วย โดยผู้เขียนหวังว่าบทความนี้จะเป็นความรู้เบื้องต้นให้กับผู้ที่สนใจทำวิจัยในสาขานี้ต่อไป

คำสำคัญ : อสมการฮาร์แนค การประมาณค่าเคอร์เนลความร้อน สมบัติทวีคูณ อสมการปวงกาเร

Abstract

Heat equations are used to explain the distribution of heat over spaces and time. Typically, solutions of heat equations can be written as integrations of their initial heat profiles and heat kernels. Thus, heat kernel behavior directly influences the behavior of solutions of heat equations and vice versa. This article will review one such relation: the Gaussian behavior of heat kernel and the (parabolic) Harnack inequality. This article will also discuss how these two properties relate to the geometric properties such as doubling property and Poincare inequality of the underlying spaces. Hopefully, this will serve as an introduction to those interested in working in this field.

Keywords : Harnack inequality, heat kernel estimates, doubling property, Poincare inequality

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Introduction

The classical heat equation is given by $\partial_t u = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^2 u$, where the function $u: [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying the above equation is called a solution of the classical heat equation. Typically, $u(t,x)$ is interpreted as the temperature at point $x \in \mathbb{R}^n$ and at time $t \geq 0$ i.e. the solution u represents the evolution of temperature over space and time. It is well-known that solution u of the classical heat equation satisfies

$$u(t,x) = \int_{\mathbb{R}^n} u(0,y) \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} \, dy$$

provided that the initial heat profile $u(0,\cdot): \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies some regularity assumptions. The function $p(t,x,y) = e^{-\frac{|x-y|^2}{4t}} / (4\pi t)^{n/2}$ is called the heat kernel associated to the Laplacian $\Delta = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^2$.

General heat equations are given by replacing the Laplacian with other heat operators. One question any researcher might ask is that whether or not the heat kernel associated to that heat operator exhibits the same behavior as that of the heat kernel associated to the Laplacian. Less than sixty years ago, Nash (Nash, 1958), Moser (Moser, 1961, 1964, 1967), and Aronson (Aronson, 1967, 1968, 1971) independently worked on this problem which can be deduced that the result holds for uniformly elliptic operators on $\mathbb{R}^n$ i.e. for heat operators of the form $\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j}$ such that $a_{ij} = a_{ji}: \mathbb{R}^n \rightarrow \mathbb{R}$ is smooth for all $i,j = 1, ..., n$ and there exists $\lambda \in (0,1)$ for which $\lambda \sum_{i=1}^{n} y_i^2 \leq a_{ij}(x) y_i y_j \leq \lambda^{-1} \sum_{i=1}^{n} y_i^2$ for all $x, y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Since then the result has been progressed in many directions. This article will focus on the results related to the perturbation of the heat equation using Dirichlet forms approach.

Dirichlet Forms

To simplify the analysis, the author assumes that the topological space X is locally compact but non-compact, second countable and Hausdorff, the Borel measure ν on X always has full support, and the Dirichlet form $(E,D(E))$ on $L^2(X, \nu)$ is regular, strongly local, and admits the carre du champ operator.

Definition 1. *A Dirichlet form* is a pair $(E,D(E))$ where $E: D(E) \times D(E) \rightarrow \mathbb{R}$ is a closed, positive symmetric, densely defined, bilinear form on $L^2(X, \nu)$ with the following property: for any $f \in D(E)$, the function $g = (f \vee 0) \wedge 1 \in D(E)$ and $E(g,g) \leq E(f,f)$.

The set $D(E) \subset L^2(X,\nu)$ equipped with the inner product $<\cdot,\cdot>_{L^2(X,\nu)} + E(\cdot,\cdot)$ is a Hilbert space, called a *Dirichlet space*, and its norm associated to its inner product is referred to as the *Dirichlet norm*.

The regularity assumption means the intersection of $D(E)$ and $C_c(X)$, the set of real-valued continuous functions with compact support on X, is dense in $C_c(X)$ under the uniform metric. This is necessary. The domain of a Dirichlet form must contain enough continuous functions, otherwise it will not reflect analytic properties of the underlying spaces. See (Oshima *et al*, 2010) for further treatment of regularity of Dirichlet forms.

A Dirichlet form $(E,D(E))$ is strongly local if $E(f,g) = 0$ for any $f, g \in D(E)$ with $(f-a)g = 0$ for some constant $a \in \mathbb{R}$. The fact is strong locality is equivalent to Leibniz rule and chain rule (Sturm, 1994). Hence, strong locality informally implies that the Dirichlet form's behavior is that of differential operators.

Note that any regular Dirichlet form $(E,D(E))$ is uniquely associated to an energy measure[2] Γ which might not be absolutely continuous with respected to the volume measure ν. If it happens that $d\Gamma(f,g)$ is always absolutely continuous with respected to ν, then that Dirichlet form is said to admit the carre du champ operator $(f,g) \mapsto \frac{d\Gamma(f,g)}{d\nu}$, $f,g \in D(E)$. Sufficient conditions for the existence of the care du champ operator is discussed in Bouleau and Hirsch's book

(Bouleau and Hirsch, 1991) and will not be repeated here.

Assuming the existence of the carre du champ operator might seem strong. The carre du champ operators, however, provide a way to define the distance

$$\rho(x,y) = \sup\left\{|f(x)-f(y)| \;\middle|\; f \in D(E) \; and \; \frac{d\Gamma(f,f)}{dv} \le 1\right\}, \forall x,y \in X$$

called the intrinsic distance on X. This ties the Dirichlet form to the geometry of the underlying space X. In order to define the intrinsic distance, it is necessary that $d\Gamma(f,f)$ is absolutely continuous with respect to the volume measure ν for functions f in a sufficiently large subset of the domain $D(E)$. Therefore, the existence of the carre du champ operator is natural.

A common assumption of ρ in the heat kernel analysis is that it must be a complete metric and it must metrise the topology of X. Assuming ρ metrises the topology of X is necessary but assume it is a complete metric is redundant since it is always possible to replace X with its completion. Under these assumptions, (X,ρ) is a geodesic space i.e. the distance between any two points in X is actually the length of a shortest path between those two points, see (Sturm, 1995a) and (Hirsch, 2003) for the proof of this fact.

Heat Equation

Let $(L,D(L))$ be a heat operator i.e. a nonpositive, densely defined, self-adjoint operator on $L^2(X,\nu)$. It is well-known (Oshima *et al*, 2010) that there is one and only one Dirichlet form $(E,D(E))$ on $L^2(X,\nu)$ such that $D(L) \subset D(E)$ is dense in $D(E)$ under the Dirichlet norm, and $E(f,g) = -\langle f, Lg \rangle$ for all $f,g \in D(L)$. Therefore, studying heat operators and Dirichlet forms are complementary.

A heat equation with heat operator L is nothing but a generalization of the classical heat equation with the Laplacian Δ. Informally, a solution of the heat equation associated to L is a function u such that $\partial_t u = Lu$. This can be interpreted as $\int \langle \partial_t u, v \rangle dt = \int \langle Lu,v \rangle dt = -\int E(u,v)dt$ for all test functions v. However, what are test functions?

Given a Hilbert space $\mathbb{H}$ and an open interval $I \subset \mathbb{R}$ denote $L^2(I \to \mathbb{H})$ the Hilbert space of all measurable functions $u : I \to \mathbb{H}$ with finite norm

$$\|u\| = \left(\int_I \|u(t)\|^2 \, dt\right)^{\frac{1}{2}} < \infty.$$

Let $W^1(I \to \mathbb{H})$ be the set of all functions $u \in L^2(I \to \mathbb{H})$ whose distributional derivative u' can be represented as a function in $L^2(I \to \mathbb{H})$. Equip $W^1(I \to \mathbb{H})$ with the norm

$$\|u\| = \left(\int_I \|u(t)\|^2 \, dt + \int_I \|u'(t)\|^2 \, dt\right)^{\frac{1}{2}}.$$

makes $W^1(I \to \mathbb{H})$ a Hilbert space.

Note that $D(E) \subset L^2(X,\nu)$ so $D(E) \subset L^2(X,\nu) \subset D(E)^*$, the L^2-dual of $D(E)$. Therefore, $F(I \times X) = L^2(I \to D(E)) \cap W^1(I \to D(E)^*)$ is a well-defined object. Denote $F_c(I \times X) = \{u \in F(I \times X) \mid u(t,\cdot)$ has a compact support for a.e. $t \in I\}$ and also denote $F_{loc}(I \times X)$ the set of all functions $u : I \times X \to \mathbb{R}$ such that for any relatively compact open subset $J \times V$ of $I \times X$, there exists a function $u_V \in F(I \times X)$ satisfying $u = u_V$ a.e. on $J \times V$.

In order to view a solution of heat equations as a function from I to $D(E)$, any function $u: I \times X \to \mathbb{R}$ will be viewed as a function $u: I \to (X \to \mathbb{R})$.

Definition 2. Let I be an open time interval and L be a heat operator associated to a strongly local regular Dirichlet form $(E,D(E))$ on $L^2(X,\nu)$. A function $u : I \times X \to \mathbb{R}$ is a (*local*) weak solution of the heat equation $\partial_t u = Lu$ if

a) $u \in F_{loc}(I \times X)$, and

b) for any open interval J relatively compact in I and any $\phi \in F_c(I \times X)$,

[2] This means $E(f,g) = \int d\Gamma(f,g)$ for all $f,g \in D(E)$.

$$\int_J < \partial_t u, \phi >_{L^2(X,\nu)} dt + \int_J E(\phi(t,\cdot), u(t,\cdot))$$

$$dt = 0.$$

Simple examples of weak solutions in the sense introduced above are functions $u(t,\cdot) = e^{-tL} f$ for $t \in I \subset (0,\infty)$, where I is any bounded interval, and $f \in L^2(X,\nu)$. More interesting examples are given in (Aronson, 1968), see also (Gyrya & Saloff-Coste, 2011).

Note that fixing an open set V and replacing $D(E)$with the closure of the set $\{f \in D(E) \mid f$ has compact support in $V\}$ in the above definition yields the definition of local solutions (in V) of heat equations.

Definition 3. An associated heat kernel of a strongly local regular Dirichlet form $(E,D(E))$ on $L^2(X, \nu)$ is a nonnegative function $p: (0,\infty) \times X \times X \to \mathbb{R}$ such that any nonnegative solution u of the heat equation associated to $(E,D(E))$ satisfies $u(t,x) = \int u(0,y)\, p(t,x,y) d\nu(y)$ for all $t > 0$ and $x \in X$.

The author ends this section with the definition of the (uniform) parabolic Harnack inequality.

Definition 4. A strongly local regular Dirichlet form $(E,D(E))$ associated to a heat operator L on $L^2(X, \nu)$ satisfies (*uniform*) *parabolic Harnack inequality* if there exists a constant $H_0 \geq 1$ such that for any $x \in X, r > 0$ and any non-negative weak solution u of the heat equation $\partial_t u = Lu$ on $(0,r^2) \times B(x,r)$,

$$\sup_{Q_-} u \leq H_0 \inf_{Q_+} u$$

where $Q_- = \left(\frac{r^2}{4}, \frac{r^2}{2}\right) \times B(x,\frac{r}{2})$, $Q_+ = \left(\frac{3r^2}{4}, r^2\right) \times B(x,\frac{r}{2})$ and both supremum and infimum are computed up to sets of measure zero.

Informally, a heat operator satisfies parabolic Harnack inequality if its solutions, further away from the boundary, are roughly constant. One consequence of uniform parabolic Harnack inequality is that such solutions always admit continuous representative so one may assume that they are continuous. For further information about Harnack inequality including its variance and history, see (Kassman, 2007).

Laplacian

A classic example of heat equations is the classical heat equation with the Laplacian $\Delta := \sum_{i=1}^n \left(\frac{\partial}{\partial x_i}\right)^2$ on $\mathbb{R}^n$ as the heat operator and the Lebesgue measure as its underlying volume measure. Its associated Dirichlet form is defined by

$$E(f,g) := \int_{\mathbb{R}^n} \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i}$$

with the domain $D(E) = H^1(\mathbb{R}^n) = W^{1,2}(\mathbb{R}^n)$, the L^2-Sobolev space of order one. The Dirichlet norm is nothing but the Sobolev norm

$$\|f\| := \int_{\mathbb{R}^n} f^2 + \int_{\mathbb{R}^n} \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\right)^2, \forall f \in H^1(\mathbb{R}^n)$$

and the intrinsic distance is nothing but the Euclidean distance on $\mathbb{R}^n$.

The heat kernel $p: (0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ associated to Δ is defined by

$$p(t,x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}$$

which will be called, for the rest of this work, the classical heat kernel.

Denote G_{σ^2} the Gaussian kernel i.e. the density of the Gaussian (Normal) distribution with mean zero and variance σ^2. Then $p(t,x,y) = G_{2t^2}(x-y)$ i.e. $p(t,\cdot,\cdot)$ is a probability density of a random variable. Actually, p is the transition density of the Brownian motion on $\mathbb{R}^n$.

In general, any heat operator is a Markov operator of some Markov processes and the heat kernel is the transition density of the corresponded Markov Processes (Chen & Fukushima, 2012). Therefore, it is possible to study Markov processes via heat equations. However, that subject will not be discussed here.

One possible question is that whether or not heat kernels associated to other heat operators behaves similarly to the Gaussian kernel.

Gaussian Behavior

In the previous section, the classical heat kernel depends on two objects, one is the intrinsic distance i.e. the Euclidean distance on $\mathbb{R}^n$, and another is the dimension n of the underlying space $\mathbb{R}^n$. It is interesting to know whether this behavior happens for other heat kernels. Consider a heat operator $L = c\Delta$ where $c > 0$ is a fixed constant. It is easy to see that u is a (weak) solution of the heat equation associated to L if and only if u solves $\frac{\partial u}{\partial(ct)} = \Delta u$. Therefore, the heat kernel associated to L is $p(t,x,y) = \frac{1}{(4\pi ct)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4ct}}$, which is again Gaussian. As a generalization of this property, one might instead ask whether a heat kernel has Gaussian upper and lower bound. Nash (Nash, 1958), Moser (Moser, 1961, 1964, 1967), and Aronson (Aronson, 1967, 1968, 1971) concluded that this is true for uniformly elliptic operator. Later on, Li and Yao(Li & Yao, 1986) extended this result to Laplacians on manifolds with nonnegative Ricci curvature. However, Li and Yao also showed that the term t^n is no longer correct and should be replaced with the volume measure. The following definition gives the current form of Gaussian behavior of heat kernel.

Definition 5. A heat kernel p is said to have *Gaussian upper bound* if there exist constants $c_1, c_2 > 0$ such that for all x, y in the underlying space and $t > 0$,

$$p(t,x,y) \le c_1 \left(\frac{1}{\sqrt{\nu(B(x,\sqrt{t}))\nu(B(y,\sqrt{t}))}} \right) e^{-c_2 \frac{\rho(x,y)}{t}}.$$

A heat kernel p is said to have *Gaussian lower bound* if there exist constants $c_3, c_4 > 0$ such that for all x, y in the underlying space and $t > 0$,

$$p(t,x,y) \ge c_3 \left(\frac{1}{\sqrt{\nu(B(x,\sqrt{t}))\nu(B(y,\sqrt{t}))}} \right) e^{-c_4 \frac{\rho(x,y)}{t}}.$$

Here ν is the volume measure and ρ is the intrinsic distance.

A heat kernel is said to satisfy *Gaussian estimates* if it has both Gaussian upper bound and Gaussian lower bound. A heat operator or a Dirichlet form is said to satisfy *Gaussian estimates if* its associated heat kernel exists and satisfies Gaussian estimates.

For heat operators in Euclidean space $\mathbb{R}^n$ with the Lebesgue measure as their volume measure ν, $\nu\left(B\left(x,\sqrt{t}\right)\right) = ct^{\frac{n}{2}}$ for all $x \in \mathbb{R}^d$ for some fixed constant $c > 0$. This reduced to the same type of estimates found in the work of Nash (Nash, 1958), Moser (Moser, 1961, 1964, 1967), and Aronson (Aronson, 1967, 1968, 1971). As a generalization of Euclidean dimensions, one might be tempted to define the (local) dimension of the underlying space X as $\lim_{t\to 0} \frac{\ln \nu(B(x,t))}{\ln t}$ or any other terms of similar nature. However, this limit usually does not exist regardless of whether the heat kernel has Gaussian estimates. Even when the limit is replaced by *limsup* or *liminf*, it still possible that the limit as t converges to zero is not the same as the limit as t converges to infinity. In other words, there is no universal definition of dimension in general. Nevertheless, Gaussian estimates still imply a property related to the growth of the volume measure called doubling property (Sturm, 1996).

Doubling Property

Actually, doubling property is a property of the underlying spaces and not of the Dirichlet forms. It has been studied as a subject by itself. The book of Heinomen (Heinomen, 2001) is a good starting point. There are two closely related versions of doubling property, one is for the space to be doubling, and another is for the volume measure of that space to be doubling. It turns out that the latter implies the former while the former implies the existence of a Borel measure which is doubling (Luukkainen & Saksman, 1998). Therefore, only doubling property of volume measures will be discussed here.

Definition 6. A non-zero Borel measure ν on a metric space (X,d) is *doubling* if there exists a constant $C_D > 0$ such that for any $x \in X$ and $r > 0$, $\nu\big(B(x,2r)\big) \leq C_D \nu(B(x,r))$.

Grigor'yan and Saloff-Coste (Grigor'yan & Saloff-Coste, 2005) extended the concept of doubling property to that of remote and S-anchored balls. They showed that doubling property of remote and S-anchored balls together with another property called volume comparison condition imply doubling property for all balls provided that the set $S \subset X$ is fully accessible. Using this fact when S is singleton, Grigor'yan and Saloff-Coste proved doubling property of weighted measures with finite unbounded weight functions. Moshimi and Tesei (Moshimi & Tesei, 2007) also used this fact to show doubling property of weighted Lebesgue measures with unique-singularity weight functions. To be precise, they proved doubling property for measures $|x|^{-\lambda}dx$ where λ is a positive number less than the dimension of the underlying Euclidean space. Later, the fully accessible assumption was weaken to that of k-skew condition and the unique-singularity assumption was removed yielding the following result. [3]

Theorem 1. (Tasena, 2011) *Let ν be a doubling measure on a metric space (X,d), $a: [0,\infty) \to (0,\infty]$, $S \subset X$ is closed and null, and $d\mu = a\big(d(\cdot,S)\big)d\nu$. Assume the following conditions hold.*

a) The set S satisfies k-skew condition *i.e. for any $o \in S$ and $r > 0$ there exists $x \in X$ such that*

$$kr \leq d(x,S) \leq d(x,o) \leq r.$$

b) The function a is remotely constant *i.e. there exists a constant $c_a > 0$ such that for any $r > 0$,*

$$sup_{[r,3r]}\, a \ \leq inf_{[r,3r]}\, a.$$

Then μ is doubling if and only if there exists a constant $c > 0$ such that for any $o \in S$ and *$r > 0$,*

$$\mu\big(B(o,r)\big) \leq ca(r)\nu\big(B(o,r)\big).$$

Examples of k-skew subsets of Euclidean spaces include proper subspaces, spheres, lattices with positive codimension, cylinders, etc. Examples of remotely constant function include positive rational functions, polynomial growth and subpolynomial growth functions such as functions with logarithmic growth rates. For further details, examples, as well as methods to check whether a weighted measure of this type satisfies doubling property, see (Tasena, 2011).

One interesting question is to classify weight functions for which their corresponding weighted measures satisfy doubling property.

Open Problem. Let ν be a doubling measure on a metric space (X,d). Find necessary and sufficient conditions of $h: X \to [0,\infty]$ for which the weighted measure $d\mu = hd\nu$ is doubling.

Poincare Inequality

The last concepts discussed in this article is the (uniform) Poincare inequality which states how the L^2-norm is controlled by the Dirichlet forms. Other inequalities of this type include Sobolev inequality, Nash inequality, etc. Recent survey article by Saloff-Coste (Saloff-Coste, 2011) discussed these concepts as well as their relationships.

Unlike doubling property, the Poincare inequality is a property of Dirichlet forms.

Definition 7. Let $(E,D(E))$ be a strongly local regular Dirichlet form on $L^2(X,\ \nu)$ with associated energy measure Γ. Then $(E,D(E))$ is said to satisfy *weak (uniform) Poincare inequality* if for some constant $k \geq 1$ and $C_P > 0$,

$$\min_{\xi\in\mathbb{R}} \int_{B(x,r)} |f-\xi|^2\ d\nu \leq C_P r^2 \int_{B(x,kr)} d\Gamma(f,f),$$

[3] Note that fully accessibility is equivalent to 1-skew condition.

$$\forall f \in D(E), x \in X, r > 0.$$

If $k = 1$, then $(E,D(E))$ is said to satisfy *(uniform) Poincare inequality*.

Under doubling property, weak Poincare inequality and Poincare inequality are equivalent (Saloff-Coste, 2002).

Grigor'yan and Saloff-Coste (Grigor'yan & Saloff-Coste, 2005) extended Poincare inequality to that of remote and S-anchored balls, where $S \subset X$ is a fully accessible set. They used this fact to prove Poincare inequality on weighted Dirichlet spaces when S is a singleton and the geodesic space X satisfies a mild additional assumption called relatively connected annuli. Their weight functions include strictly positive unbounded real-valued functions with no singularity. Later, Moshimi and Tesei(Moshimi & Tesei, 2007) extended Grigor'yan and Saloff-Costes' work to include weight functions $x \in \mathbb{R}^n \mapsto |x|^{-\lambda}$ where $0 < \lambda < n.$ Nevertheless, the same technique can be applied to other weight functions with unique singularity.

In 2011, Tasena (Tasena, 2011) extended these results to include weight functions with multiple singularities. He proved Poincare inequality under the assumptions that the weight functions are of the form $a(d(\cdot,S))$, where S is the singularity set and a is a nonincreasing remotely constant function. The singularity set S must also satisfy an additional assumption called k-accessibility.[4] This is a stronger version of k-skew condition mentioned previously.

Definition 8. A closed subset S of a geodesic space (X,d) is said to be k-accessible if it satisfies k'-skew condition for some k'>k and for any $o \in S$ and $r > 0$, the set

$$\{x \in X \mid kd(x,o) \leq d(x,S) \leq d(x,o) \leq r\}$$

is path-connected.

Theorem 2. (Tasena, 2011) *Let* $(E,D(E))$ *be a strongly local regular Dirichlet form on* $L^2(X, \nu)$ *with associated energy measure* Γ, $S \subset X$ *be a k-accessible null set,* $h = a(d(\cdot,S)) > 0$, and $d\mu = hd\nu$. *Define*

$$E^h(f,g) := \int hd\Gamma(f,g)$$

for any $f,g \in \mathcal{D} := \{e \in D(E) | E^h(e,e) < \infty\}$. *Then* $(E^h,\mathcal{D})$ *extended to a strongly local regular Dirichlet form* $(E^h,D(E^h))$, *called* the weighted Dirichlet form, *on* $L^2(X,\mu)$.

Moreover, if $(E,D(E))$ *satisfies Poincare inequality,* ν *satisfies doubling property, and a is a nonincreasing remotely constant function, then* $(E^h,D(E^h))$ *satisfies Poincare inequality if and only if the measure* μ *is doubling*.

Sturm's Result

Sturm (Sturm, 1994, 1995b, 1996) showed in a series of articles that Gaussian estimates, parabolic Harnack inequality, Poincare inequality, and doubling property are closely related.

Theorem 3. (Sturm, 1996) *Let* $(E,D(E))$ *be a strongly local regular Dirichlet form on* $L^2(X, \nu)$. *Then the following statements are equivalent.*

a) The Dirichlet form $(E,D(E))$ *satisfies parabolic Harnack inequality.*

b) The Dirichlet form $(E,D(E))$ *satisfies Gaussian estimates.*

c) The Dirichlet form $(E,D(E))$ *satisfies Poincare inequality and the volume measure* ν *satisfies doubling property.*

The fact that a) and b) are equivalent is perhaps less surprising than that they are equivalent to c). Since solutions of heat equations are integrals of the multiplication between heat kernels and the initial heat profiles, the behavior of heat kernels will directly influence the behavior of solutions of heat equations. On the contrary, heat kernels can be viewed as solutions of heat equations with Dirac measures as initial heat profiles, so parabolic Harnack inequality will force the heat kernels to satisfy Gaussian estimates. The Poincare inequality and

[4] In case of singleton S, fully accessibility and relatively connected annuli together implies k-accessibility. The converse is not true, however.

the doubling property, on the other hand, have almost nothing to do with heat equations. Yet these properties directly influence both heat kernels and solutions of heat equations. As an example, Tasena's result can be translated as follows.

Theorem 4. *Let* $(E, D(E))$ *be a strongly local regular Dirichlet form on* $L^2(X, \nu)$ *with associated energy measure* Γ, $S \subset X$ *is a* ρ*-accessible null set,* $h = a(d(\cdot, S)) > 0$, *and* $d\mu = h d\nu$.

If $(E, D(E))$ *satisfies parabolic Harnack inequality, and* a *is a nonincreasing remotely constant function, then the weighted Dirichlet space* $(E^h, D(E^h))$ *satisfies parabolic Harnack inequality if and only if there exists a constant* $c > 0$ *such that for any* $o \in S$ *and* $r > 0$, $\mu(B(o, r)) \leq c a(r) \nu(B(o, r))$.

Conclusion

In this article, the author introduces heat equations on general metric spaces using Dirichlet form approach. This is a generalization of an approach using Sobolev spaces and weakly differentiable functions. The author also discusses when heat kernels will have a nice behavior similar to the classical heat kernel. This is of course by no means a complete list of references. The author hopes that this will, however, provide an introduction to this field.

There are several ways to perturbed Dirichlet forms. Here the author introduces weighted Dirichlet spaces. The original work of Nash (Nash, 1958), Moser (Moser, 1961, 1964, 1967), and Aronson (Aronson, 1967, 1968, 1971) can also be considered results in this category. Over the years, many improvements have been made, many assumptions have been weaken. Yet, many questions remain unanswered. For examples, no one knows sufficient and necessary conditions for weighted measures to be doubling, or for weighted Dirichlet spaces to satisfy Poincare inequality. Even when the singularity set is k-accessible, no one knows the necessary and sufficient conditions of the weight functions for which the weighted Dirichlet spaces satisfy the parabolic Harnack inequality. Even whether k-accessibility is the right condition to assume on singularity sets is unknown. All these require further investigation into the subject so the author strongly believes this field will remain fruitful in years to come.

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