บางสัจพจน์การแยกในปริภูมิเชิงทอพอโลยีแบบเรียบสามัญ

Some Separation Axioms in Ordinary Smooth Topological Spaces

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Abstract

In this paper, we determine some type of the separation axioms of ordinary smooth topological spaces and we also study some properties of ordinary smooth topological spaces which are introduced by our axioms.

Keywords : ordinary smooth topological space, ordinary smooth subspace, separation axioms, continuous function, homeomorphism

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Introduction

The concepts of ordinary smooth topology on a non-empty set X was first introduced by Pyung, Byeong, and Kul in (Pyung, Byeong & Kul, 2012) as a mapping $\tau: 2^X \to I$ with the properties.

(i) $\tau(X) = \tau(\emptyset) = 1$,

(ii) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$ for all $A, B \in 2^X$,

(iii) $\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \tau(A_\alpha)$ for all $\{A_\alpha\} \subseteq 2^X$

where 2^X is the power set of X and I be the closed interval [0, 1]. The pair (X, τ) is called an *ordinary smooth topological space* (briefly, *osts*). The closure subsets, the interior subsets and compactness of an ordinary smooth topological spaces were studied in (Jeong, Pyung & Kul, 2013) and (Pyung, Byeong & Kul, 2012).

In this paper, we have to define the axioms which can be classed in the classes of ordinary smooth topological spaces.

The operators on X which induced by the ordinary smooth topologies τ are defined as follows.

Definition 1. (Jeong, Pyung & Kul, 2013) Let (X, τ) be an osts and let $A \in 2^X$. Then *ordinary smooth closure* and *ordinary smooth interior* of A in X are defined by

$$\bar{A} = \bigcap\{F \in 2^X : A \subseteq F \text{ and } \tau(F^c) > 0\} \text{ and}$$

$$A^\circ = \bigcup\{U \in 2^X : U \subseteq A \text{ and } \tau(U) > 0\}, \text{ respectively.}$$

In (Pyung, Byeong & Kul, 2012), it the morphisms of ordinary smooth topological spaces is defined as follows.

Definition 2. (Pyung, Byeong & Kul, 2012) Let $\tau_1 \in OST(X)$ and let $\tau_2 \in OST(Y)$. Then a mapping $f: X \longrightarrow Y$ is said to be *ordinary smooth continuous* (resp. *ordinary smooth open and ordinary smooth closed*) if $\tau_2(A) \leq \tau_1(f^{-1}(A))$, (resp. $\tau_1(A) \leq \tau_2(f(A))$, $\tau_1(A) \leq \tau_2(f(A))$) for all $A \in 2^Y$(resp. for all $A \in 2^X$). A mapping f is said to be *ordinary smooth homeomorphism* if f is a bijective and ordinary smooth continuous and f^{-1} is an ordinary smooth continuous.

Main Results

In this section, we will introduce the notions of OT_0, OT_1 and OT_2-spaces and investigate some of their properties.

For an osts (X, τ), define the family $S(\tau) = \{A \in 2^X : \tau(A) > 0\}$ and $S(\tau)$ will be called the *support* of τ.

Definition 3. Let (X, τ) be ordinary smooth topological space. Then OT_0, OT_1 and OT_2-spaces are defined as follows:

(i) OT_0-space if and only if for each $x, y \in X$ with $x \neq y$, there exists $U \in S(\tau)$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

(ii) OT_1-space if and only if for each $x, y \in X$ with $x \neq y$, there exist $U, V \in S(\tau)$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

(iii) OT_2-space if and only if for each $x, y \in X$ with $x \neq y$, there exist $U, V \in S(\tau)$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Example 4. Let $X = \{1,2\}$ and we define a mapping $\tau: 2^X \to I$ as follows:

$$\tau(X) = \tau(\emptyset) = 1,\ \tau(\{1\}) = 0.5 \text{ and } \tau(\{2\}) = 0.$$

Clearly, (X, τ) is an osts. Then (X, τ) is an OT_0-space, since $\{1\} \in S(\tau)$, $1 \in \{1\}$ and $1 \notin \{2\}$.

Example 5. Let X be infinite set. We define a mapping $\tau: 2^X \to I$ as follows:

$$\tau(A) = \begin{cases} 1, & \text{if } A = 0 \text{ } or \text{ } A^c \text{ is finite,} \\ 0, & \text{otherwise,} \end{cases}$$

for each $A \in 2^X$.

Clearly, (X, τ) is an osts. Let consider $x, y \in X$ with $x \neq y$. Since $y \notin \{x\}$, $y \in X \backslash \{x\}$ and $X \backslash \{x\} \in S(\tau)$. Similarly, $x \notin \{y\}$, $x \in X \backslash \{y\}$ and $X \backslash \{y\} \in S(\tau)$. It follows that (X, τ) is a OT_1-space.

Example 6. Let X be a nonempty set. We define a mapping $\tau: 2^X \to I$ as follows:

$$\tau(A) = 1, \text{ for each } A \in 2^X.$$

Then pair (X, τ) is called an *ordinary smooth discrete topological space* on X. For each $x, y \in X$ with $x \neq y$. Since $\tau(\{x\}) = 1$ and $\tau(\{y\}) = 1$, $\{x\}, \{y\} \in S(\tau)$ and $\{x\} \cap \{y\} = \emptyset$. Therefore (X, τ) is a OT_2-space.

Remark 7. If an osts (X, τ) is a OT_1-space (resp. OT_2-space), then (X, τ) is a OT_0-space (resp. OT_1-space).

The converse of remark 7. is not true, for instance (X, τ) in Example 4 (example 5).

The following results therefore follows directly from the definition of ordinary smooth closure.

Lemma 8. ((Jeong, Pyung & Kul, 2013), Proposition 3.2 (i) and Proposition 3.6 (b)) Let (X, τ) be an osts and let $A, B \in 2^X$

(i) If $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$.

(ii) If $\tau(A^c) > 0$, then $A = \bar{A}$.

On we gives a very simple characterization of OT_0 and OT_1-spaces.

Theorem 9. An osts (X, τ) is a OT_0-space if and only if for every $x, y \in X$ such that $x \neq y$, we have $\overline{\{x\}} \neq \overline{\{y\}}$.

Proof. Suppose that (X, τ) is an OT_0-space and let $x, y \in X$ such that $x \neq y$. By assumption we may assume that there exists $U \in S(\tau)$ such that $x \in U$, $y \notin U$. Since $\{y\} \subseteq X \backslash U$ and $\tau(U) > 0$, By Lemma 8. (i) and (ii), we get $\overline{\{y\}} \subseteq X \backslash U$. But $\{x\} \nsubseteq X \backslash U$. Therefore $\overline{\{x\}} \neq \overline{\{y\}}$.

Conversely, assume that $\overline{\{x\}} \neq \overline{\{y\}}$ for all $x, y \in X$ such that $x \neq y$. We will show that (X, τ) is a OT_0-space. By assumption, we may assume that $x \notin \overline{\{y\}}$. Then there exists $F \in 2^X$ such that $\{y\} \subseteq F$, $\tau(F^c) > 0$ and $x \notin F$. Let $U = F^c$. Since $x \notin F$ and $y \in F$, $x \in U$ and $y \notin U$. Therefore (X, τ) is a OT_0-space.

Theorem 10. An osts (X,τ) is a OT_1-space if and only if $\overline{\{x\}} = \{x\}$ for every $x \in X$.

Proof. Assume that (X,τ) is a OT_1-space. We will show that $\overline{\{x\}} = \{x\}$. Suppose that $\overline{\{x\}}\backslash\{x\} \neq \emptyset$. Then there exists $y \in \overline{\{x\}}\backslash\{x\}$. Since (X,τ) is a OT_1-space, there exists $U_y \in S(\tau)$ such that $y \in U_y$ and $x \notin U_y$, so $y \notin (U_y)^c$ and $\{x\} \subseteq (U_y)^c$. But $y \in \overline{\{x\}}$, we have $y \in F$ for all F such that $\{x\} \subseteq F$ and $F^c \in S(\tau)$. Since $\{x\} \subseteq (U_y)^c$ and $U_y \in S(\tau)$, $y \in (U_y)^c$. This is a contradiction. Thus $\overline{\{x\}}\backslash\{x\} = \emptyset$. Hence $\overline{\{x\}} = \{x\}$.

Conversely, suppose that $\overline{\{x\}} = \{x\}$ for all $x \in X$. We will show that (X,τ) is a OT_1-space. Let $x, y \in X$ such that $x \neq y$. By assumption we have that $x \in X\backslash\overline{\{y\}}$, $y \in X\backslash\overline{\{x\}}$, $x \notin \overline{\{y\}}$ and $y \notin \overline{\{x\}}$. There exist $F_1, F_2 \in 2^X$ such that $\{y\} \subseteq F_1$, $\{x\} \subseteq F_2$, $\tau({F_1}^c) > 0$, $\tau({F_2}^c) > 0$ and $x \notin F_1$, $y \notin F_2$. Let $U_1 = {F_1}^c$ and $U_2 = {F_2}^c$. Then $y \in U_1$, $x \notin U_1$ and $x \in U_2$, $y \notin U_2$ where $U_1, U_2 \in S(\tau)$. Therefore (X,τ) is a OT_1-space.

The next theorem, we gives the characterization of OT_2-spaces.

Theorem 11. Let (X,τ) be an osts. Then the following conditions are equivalent:

(i) (X,τ) is an OT_2-space.

(ii) Let $p \in X$ for $q \neq p$ there exists $U \in S(\tau)$, $p \in U$ such that $q \notin \overline{U}$.

(iii) For each $p \in X$, $\cap\{\overline{U} : U \in S(\tau),\ p \in U\} = \{p\}$.

Proof. (i) $\Longrightarrow$ (ii) Let $p, q \in X$ with $q \neq p$. Since (X,τ) is an OT_2-space, there exist $U, V \in S(\tau)$ such that $p \in U$, $q \in V$ and $U \cap V = \emptyset$. Thus $U \subseteq X\backslash V$. By Lemma 8. (i) and (ii), we have $\overline{U} \subseteq \overline{X\backslash V}$ and $\overline{X\backslash V} = X\backslash V$. Since $q \in V$, then $q \notin X\backslash V$. Hence $q \notin \overline{U}$.

(ii) $\Longrightarrow$ (iii) Let $p \in X$. We will show that $\cap\{\overline{U} : U \in S(\tau),\ p \in U\} = \{p\}$. Clearly, $\{p\} \subseteq \cap\{\overline{U} : U \in S(\tau),\ p \in U\}$. Sufficient to prove that $\cap\{\overline{U} : U \in S(\tau),\ p \in U\} \subseteq \{p\}$. Let $q \in X$ with $q \neq p$. Then $q \notin \{p\}$. By (ii), there exist $U_1 \in S(\tau)$, $p \in U_1$ such that $q \notin \overline{U_1}$. Then $q \notin \cap\{\overline{U} : U \in S(\tau),\ p \in U\}$. Hence $\cap\{\overline{U} : U \in S(\tau),\ p \in U\} \subseteq \{p\}$. Therefore $\cap\{\overline{U} : U \in S(\tau),\ p \in U\} = \{p\}$.

(iii) $\Longrightarrow$ (i) Assume that $\{p\} = \cap\{\overline{U} : U \in S(\tau),\ p \in U\}$. Let $p, q \in X$ with $q \neq p$, then $q \notin \{p\} = \cap\{\overline{U} : U \in S(\tau),\ p \in U\}$. Then there exists $U_1 \in S(\tau)$ such that $p \in U_1$ and $q \notin \overline{U_1}$. Since $\overline{U_1} = \cap\{V \in 2^X : U_1 \subseteq V$ and $\tau(V^c) > 0\}$, there exists $V \in 2^X$ such that $U_1 \subseteq V$, $\tau(V^c) > 0$ and $q \notin V$. Let $K = V^c$. Since $q \notin V$, $q \in V^c = K$. Next, we will show that $K \in S(\tau)$. Since $K = V^c$, $\tau(K) > 0$. Hence $K \in S(\tau)$. Next, to show that $K \cap U_1 = \emptyset$. Since $V \cap V^c = \emptyset$ and $U_1 \subseteq V$. $K \cap U_1 = \emptyset$. Hence (X,τ) is a OT_2-space.

The following results therefore follows directly from the Proposition 5.1.(Pyung, Byeong & Kul, 2012) of ordinary smooth subspace.

Lemma 12. (Pyung, Byeong & Kul, 2012) Proposition 5.1.) Let (X,τ) be an osts and let $A \subseteq X$. We define a mapping $\tau_A : 2^A \longrightarrow I$ as follows. For each $B \in 2^A$,

$$\tau_A(B) = \bigvee\{\tau(C) : C \in 2^X \text{ and } C \cap A = B\}.$$

Then $\tau_A \in OST(A)$ and $\tau(B) \geq \tau_A(B)$. In this case, (A, τ_A) is called an *ordinary smooth subspace* of (X,τ) and τ_A is called the induced *ordinary smooth topology* on A by τ.

For the subspaces of OT_0, OT_1 and OT_2-spaces, we have the following results.

Proposition 13. Every subspace of OT_2-spaces (resp. OT_0, OT_1-spaces) is also OT_2-spaces (resp. OT_0, OT_1-spaces).

Proof. Let (X,τ) be an OT_2-space, let $(A,\ \tau_A)$ be an ordinary smooth subspace of (X,τ). For any $a_1, a_2 \in A$ such that $a_1 \neq a_2$. Since (X,τ) is a OT_2-space, we may assume that there exist $U,V \in S(\tau)$ such that $a_1 \in U$, $a_2 \in V$ and $U \cap V = \emptyset$. Let $B = U \cap A$ and $C = V \cap A$. Then,

$$\begin{aligned}\tau_A(B) &= \bigvee\{\tau(U) : U \in 2^X \text{ and } U \cap A = B\}\\ &\geq \tau(U)\\ &> 0.\end{aligned}$$

Similarly, $\tau_A(C) > 0$. Clearly, $B \cap C = \emptyset$. So $B, C \in S(\ \tau_A)$ such that $a_1 \in B$, $a_2 \in C$. Hence $(A,\ \tau_A)$ is a OT_2-space.

The following results therefore follows directly from the Definition 3.10. (Pyung, Byeong & Kul, 2012) of ordinary smooth homeomorphism.

Lemma 14.((Pyung, Byeong & Kul, 2012) Theorem 3.11.) Let $(X,\ \tau_1)$ and $(Y,\ \tau_2)$ be two osts and let $f: X \to Y$ be a mapping. Then the following are equivalent:

(i) f is an ordinary smooth homeomorphism.

(ii) f is ordinary smooth open and ordinary smooth continuous.

(iii) f is ordinary smooth closed and ordinary smooth continuous.

Lemma 15.((Pyung, Byeong & Kul, 2012) Theorem 3.3. (a)) Let $(X,\ \tau_1)$ and $(Y,\ \tau_2)$ be two osts and let $f: X \to Y$ is said to be ordinary smooth continuous if $\tau_2(A^c) \leq \ \tau_1(f^{-1}(A))$ for all $A \in 2^Y$.

The following results are the properties of OT_0, OT_1 and OT_2-spaces under some kinds of ordinary smooth maps.

Proposition 16. Let $(X,\ \tau_1)$ and $(Y,\ \tau_2)$ be two osts and let $f: X \to Y$ be an ordinary smooth homeomorphism. Then $(X,\ \tau_1)$ is an OT_2-space (resp. OT_0 and OT_1-space) if and only if $(Y,\ \tau_2)$ is an OT_2-space (resp. OT_0 and OT_1-space).

Proof. Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since f is a bijective, there are $x_1, x_2 \in X$ such that $y_1 = f(x_1)$, $y_2 = f(x_2)$, and $x_1 \neq x_2$. Since $(X,\ \tau_1)$ is an OT_2-space, there exist $U,V \in S(\ \tau_1)$ such that $x_1 \in U$, $x_2 \in V$ and $U \cap V = \emptyset$. Since f is an ordinary smooth open, it follows that

$$\tau_2\big(f(U)\big) \geq \ \tau_1(U) > 0 \text{ and } \tau_2\big(f(V)\big) \geq \ \tau_1(V) > 0.$$

Thus $f(U), f(V) \in S(\ \tau_2)$. Since f is a bijective, $y_1 \in f(U)$, $y_2 \in f(V)$ and $f(U) \cap f(V) = \emptyset$. Hence $(Y,\ \tau_2)$ is a OT_2-space.

Conversely, let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Since f is a bijective, there are $y_1, y_2 \in Y$ such that $x_1 = f(y_1)$, $x_2 = f(y_2)$, and $y_1 \neq y_2$. Since (Y, τ_2) is a OT_2-space, there exist $U, V \in S(\tau_2)$ such that $y_1 \in U$, $y_2 \in V$ and $U \cap V = \emptyset$. Since f is an ordinary smooth continuous, it follows that

$$\tau_1\big(f^{-1}(U)\big) \geq \tau_2(U) > 0 \text{ and } \tau_1\big(f^{-1}(V)\big) \geq \tau_2(V) > 0.$$

Thus $f^{-1}(U), f^{-1}(V) \in S(\tau_1)$. Since f is a bijective, $x_1 \in f^{-1}(U)$, $x_2 \in f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence (X, τ_1) is an OT_2-space.

Proposition 17. Let $f: X \to Y$ be an injective, ordinary smooth continuous map with respect to the ordinary smooth topologies τ_1 and τ_2 respectively. If (Y, τ_2) is an OT_2-space (resp. OT_0 and OT_1-space), so is a (X, τ_1).

Proof. Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Since f is an injective, we have $f(x_1) \neq f(x_2)$. Since (Y, τ_2) is an OT_2-space, there exist $U, V \in S(\tau_2)$ such that $f(x_1) \in U$, $f(x_2) \in V$ and $U \cap V = \emptyset$. Since f is an injective and ordinary smooth continuous, it follows that

$$\tau_1\big(f^{-1}(U)\big) \geq \tau_2(U) > 0 \text{ and } \tau_1\big(f^{-1}(V)\big) \geq \tau_2(V) > 0,$$

$$f^{-1}(U) \cap f^{-1}(V) = \emptyset,$$

$$x_1 = f^{-1}(f(x_1)) \in f^{-1}(U) \text{ and } x_2 = f^{-1}(f(x_2)) \in f^{-1}(V).$$

So $f^{-1}(U), f^{-1}(V) \in S(\tau_1)$ such that $x_1 \in f^{-1}(U)$, $x_2 \in f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence (X, τ_1) is an OT_2-space.

Proposition 18. An osts (X, τ_1) is an OT_2-space (resp. OT_0 and OT_1-space). If $f: X \to Y$ is an injective and ordinary smooth open, then $(f(X), \tau_{2f(X)})$ is an OT_2-space (resp. OT_0 and OT_1-space).

Proof. Let $(f(X), \tau_{2f(X)})$ be an ordinary smooth subspace of (Y, τ_2). For any $a, b \in f(X)$ such that $a \neq b$. Since f is an injective, $f^{-1}(a) \neq f^{-1}(b)$. Since (X, τ_1) is an OT_2-space, there exist $U, V \in S(\tau_1)$ such that $f^{-1}(a) \in U$, $f^{-1}(b) \in V$ and $U \cap V = \emptyset$. Since f is an ordinary smooth open and $(f(X), \tau_{2f(X)})$ is an ordinary smooth subspace of (Y, τ_2) and $f(U), f(V) \subseteq f(X)$,

$$\tau_{2f(X)}(f(U)) \geq \tau_2\big(f(U)\big) \geq \tau_1(U) > 0,$$

$$\tau_{2f(X)}(f(V)) \geq \tau_2\big(f(V)\big) \geq \tau_1(V) > 0.$$

Thus $f(U), f(V) \in S(\tau_{2f(X)})$. Since f is an injective, $a \in f(U)$, $b \in f(V)$ and $f(U) \cap f(V) = \emptyset$. Hence, $(f(X), \tau_{2f(X)})$ is an OT_2-space.

Proposition 19. An osts (Y, τ_2) is an OT_2-space (resp. OT_0 and OT_1-space). If $f: X \to Y$ is an injective and ordinary smooth continuous, then $(f^{-1}(Y), \tau_{1f^{-1}(X)})$ is an OT_2-space (resp. OT_0 and OT_1-space).

Proof. Let $(f^{-1}(Y), \tau_{1f^{-1}(X)})$ be an ordinary smooth subspace of (X, τ_1). For any $a, b \in f^{-1}(Y)$ such that $a \neq b$. Since f is an injective, $f(a) \neq f(b)$. Since (Y, τ_2) is an OT_2-space, there exist $U, V \in S(\tau_2)$ such that $f(a) \in U$, $f(b) \in V$ and $U \cap V = \emptyset$. Since f is an ordinary smooth continuous and $(f^{-1}(Y), \tau_{1f^{-1}(X)})$ is an ordinary smooth subspace of (X, τ_1) and $f^{-1}(U), f^{-1}(V) \subseteq X$,

$$\tau_{1f^{-1}(Y)}(f^{-1}(U)) \geq \tau_1\big(f^{-1}(U)\big) \geq \tau_2(U) > 0,$$
$$\tau_{1f^{-1}(Y)}(f^{-1}(V)) \geq \tau_1\big(f^{-1}(V)\big) \geq \tau_2(V) > 0.$$

Thus $f^{-1}(U), f^{-1}(V) \in S(\tau_{1f^{-1}(Y)})$. Since f is an injective, $a \in f^{-1}(U), b \in f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V)\emptyset$. Hence $(f^{-1}(Y), \tau_{1f^{-1}(X)})$ is an OT_2-space.

Conclusions

The following diagram illustrates the relationship between the spaces discussed in this section.

OT_0-spaces $\longleftarrow$ OT_1-spaces $\longleftarrow$ OT_2-spaces

Even though, I have found several properties as of the spaces presented in this paper, there are several questions yet to be answered and it may be worth investigating in future studies. I formulate the questions as follows:

1. Are there another properties of the OT_0, OT_1 and OT_2-spaces ?
2. Are there another properties of other types of separation axioms on the ordinary smooth topological space?

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