สัจพจน์การแยกอย่างอ่อนในปริภูมิสองโครงสร้างอ่อน

Weak Separation Axioms in Bi-weak Structure Spaces

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บทคัดย่อ

ในบทความนี้ จะแนะนำปริภูมิแบบใหม่ที่ประกอบด้วยเซต X ที่ไม่ใช่เซตว่าง และโครงสร้างอ่อนบน X สองโครงสร้าง เรียกปริภูมินี้ว่า ปริภูมิสองโครงสร้างอ่อน และเรียกโดยย่อว่า ปริภูมิสอง w นอกจากนี้ยังได้ศึกษาสมบัติบางประการของเซตปิด เซตเปิด และลักษณะเฉพาะบางประการของสัจพจน์การแยกอย่างอ่อนในปริภูมินี้

คำสำคัญ : โครงสร้างอ่อน; สองโครงสร้างอ่อน; ปริภูมิสอง w-T_0; ปริภูมิสอง w-T_1; ปริภูมิสอง w-R_0

Abstract

In this article, a new space, which consists of a nonempty set X and two weak structures on X, is introduced. It is called a bi-weak structure space or briefly a bi-w space. Some properties of closed sets and open sets are studied in this space. Furthermore, some characterizations of weak separation axioms are obtained.

Keywords : weak structure, bi-weak structure, T_0-bi-w spaces, T_1-bi-w spaces, R_0-bi-w spaces.

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Introduction

The notions of minimal structures, generalized topologies and weak structures are generalizations of topologies. Minimal structures was introduced by Popa and Noiri (2002). Generalized topologies and weak structures were studied by Csa´sza´r (2002) and (2011), respectively.

The concept of a bitopological space, which is a set with two topologies, was studied by Kelly (1963). Later, Boonpok (2010), introduced bigeneralized topological spaces and biminimal structure spaces, respectively. Buadong *et al.* (2011), introduced generalized topology and minimal structure spaces or briefly GTMS spaces, which is a set equipped with a generalized topology and minimal structure. Moreover, they studied some separation axioms in GTMS spaces. Recently, Zakari (2013), introduced some generalizations of closed sets in GTMS spaces.

In this article, we study a new space which consists of a set X and two weak structures on X. We will call it a bi-weak structure space or briefly a bi-w space. Such space is a generalization of bitopological spaces, bigeneralized topological spaces, biminimal structure spaces and GTMS spaces. Also, we study some properties of closed sets and open sets in the space. Moreover, we introduce the concepts of some separation axioms and study relationships with closed sets and other types of separation axioms in bi-w spaces.

Preliminaries

In this section, we discusses about some properties of weak structure spaces including some properties of closure and interior in w-space.

Definition 2.1. [Csa´sza´r (2011)]. Let X be a nonempty set and $P(X)$ the power set of X. A subfamily w of $P(X)$ is called a *weak structure* (briefly WS) on X if $\emptyset \in w$.

By (X, w) we denote a nonempty set X with a WS w on X and it is called a w-*space*. The elements of w are called w-open sets and the complements are called w-closed sets.

Let w be a weak structure on X and $A \subset X$, the w-closure of A, denoted by $c_w(A)$ and w-interior of A denoted by $i_w(A)$. We define $c_w(A)$ as the intersection of all w-closed sets containing A and $i_w(A)$ as the union of all w-open subsets of A.

Theorem 2.2. [Csa´sza´r (2011)]. If w is a WS on X and $A, B \subset X$. Then

1. $A \subset c_w(A)$ and $i_w(A) \subset A$;
2. if $A \subset B$, then $c_w(A) \subset c_w(B)$ and $i_w(A) \subset i_w(B)$;
3. $c_w\big(c_w(A)\big) = c_w(A)$ and $i_w(i_w(A)) = i_w(A)$;
4. $c_w(X\backslash A) = X\backslash i_w(A)$ and $i_w(X\backslash A) = X\backslash c_w(A)$;
5. $x \in i_w(A)$ if and only if there is a w-open set V such that $x \in V \subset A$;
6. $x \in c_w(A)$ if and only if $V \cap A \neq \emptyset$ for any w-open set V containing x;
7. if $A \in w$, then $A = i_w(A)$ and if A is w-closed, then $A = c_w(A)$.

Results

In this section, we introduce the concept of bi-weak structure spaces and study some fundamental properties of closed sets and open sets in bi-weak structure spaces.

3.1 Bi-weak structure spaces

Definition 3.1.1. Let X be a nonempty set and w^1, w^2 be two weak structures on X. A triple (X, w^1, w^2) is called a *bi-weak structure space* (briefly *bi-w space*).

Let (X, w^1, w^2) be a bi-w space and A be a subset of X. The w-closure and w-interior of A with respect to w^j are denoted by $c_{w^j}(A)$ and $i_{w^j}(A)$, respectively, for $j \in \{1, 2\}$.

Definition 3.1.2. A subset A of a bi-weak structure space (X, w^1, w^2) is called *closed* if $A = c_{w^1}(c_{w^2}(A))$. The complement of a closed set is called *open*.

Theorem 3.1.3. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then the following are equivalent:

1. A is closed;
2. $A = c_{w^1}(A)$ and $A = c_{w^2}(A)$;
3. $A = c_{w^2}(c_{w^1}(A))$.

Proof. 1. $\Longrightarrow$ 2. Assume that A is closed. Then $A = c_{w^1}(c_{w^2}(A))$. Thus $A \subset c_{w^1}(A) \subset c_{w^1}\big(c_{w^2}(A)\big) = A$ and $A \subset c_{w^2}(A) \subset c_{w^1}\big(c_{w^2}(A)\big) = A$. Hence $A = c_{w^1}(A)$ and $A = c_{w^2}(A)$.

2. $\Longrightarrow$ 3. It is clear.

3. $\Longrightarrow$ 1. Assume that $c_{w^2}(c_{w^1}(A)) = A$. Then $c_{w^1}\Big(c_{w^2}\big(c_{w^1}(A)\big)\Big) = c_{w^1}(A)$. Thus $A \subset c_{w^1}\big(c_{w^2}(A)\big) \subset c_{w^1}\Big(c_{w^2}\big(c_{w^1}(A)\big)\Big) = c_{w^1}(A) \subset c_{w^2}\big(c_{w^1}(A)\big) = A$. Hence A is closed.

Example 3.1.4. Let $X = \{1, 2, 3\}$. Define WS w^1 and w^2 on X as follows:

$$w^1 = \{\emptyset, \{1\}, \{2\}, \{1, 3\}\} \text{ and } w^2 = \{\emptyset, \{1\}, \{3\}, \{1, 2\}\}.$$

Then (X, w^1, w^2) is a bi-w space. Put $A = \{2\}$. Since $\{2\} = c_{w^1}(\{2\})$ and $\{2\} = c_{w^2}(\{2\})$, then $\{2\}$ is closed. Thus $\{1, 3\}$ is open.

Proposition 3.1.5. Let (X, w^1, w^2) be a bi-w space and $A \subset X$. If A is both w^1-closed and w^2-closed, then A is a closed set in the bi-w space (X, w^1, w^2).

Proof. It follows from Theorem 2.2 (7) and Theorem 3.1.3.

Proposition 3.1.6. Let (X, w^1, w^2) be a bi-w space. If A_α is closed for all $\alpha \in \Lambda \neq \emptyset$, then $\bigcap_{\alpha\in\Lambda} A_\alpha$ is closed.

Proof. Assume that A_α is closed for all $\alpha \in \Lambda$. Then $c_{w^1}(c_{w^2}(A_\alpha)) = A_\alpha$ for all $\alpha \in \Lambda$. Let $\beta \in \Lambda$. Then $\bigcap_{\alpha\in\Lambda} A_\alpha \subset A_\beta$. Thus $c_{w^1}(c_{w^2}(\bigcap_{\alpha\in\Lambda} A_\alpha)) \subset c_{w^1}(c_{w^2}\big(A_\beta\big)) = A_\beta$ for all $\beta \in \Lambda$. This implies $c_{w^1}(c_{w^2}(\bigcap_{\alpha\in\Lambda} A_\alpha)) \subset \bigcap_{\alpha\in\Lambda} A_\alpha$. Since $\bigcap_{\alpha\in\Lambda} A_\alpha \subset c_{w^1}(c_{w^2}(\bigcap_{\alpha\in\Lambda} A_\alpha))$, we obtain $c_{w^1}(c_{w^2}(\bigcap_{\alpha\in\Lambda} A_\alpha)) = \bigcap_{\alpha\in\Lambda} A_\alpha$. Therefore $\bigcap_{\alpha\in\Lambda} A_\alpha$ is closed.

Remark 3.1.7. The union of two closed sets is not a closed set in general as can be seen from the following example.

Example 3.1.8. Let $X = \{1,2,3\}$. Define WS w^1 and w^2 on X as follows:

$$w^1 = \{\emptyset, \{1,3\}, \{2,3\}, X\} \text{ and } w^2 = \{\emptyset, \{1\}, \{1,3\}, \{2,3\}, X\}.$$

Then $\{1\}$ and $\{2\}$ are closed but $\{1\} \cup \{2\} = \{1,2\}$ is not closed.

Theorem 3.1.9. Let (X, w^1, w^2) be a bi-w space and A be a subset of X. Then the following are equivalent:

1. A is open;
2. $A = i_{w^1}(i_{w^2}(A))$;
3. $A = i_{w^1}(A)$ and $A = i_{w^2}(A)$;
4. $A = i_{w^2}(i_{w^1}(A))$.

Proof. It follows from Theorem 3.1.3 and Theorem 2.2.

Proposition 3.1.10. Let (X, w^1, w^2) be a bi-w space. If A_α is open for all $\alpha \in \Lambda \neq \emptyset$, then $\bigcup_{\alpha\in\Lambda} A_\alpha$ is open.

Proof. Assume that A_α is open for all $\alpha \in \Lambda$. Then $X \setminus A_\alpha$ is closed. Thus $X \setminus \bigcup_{\alpha\in\Lambda} A_\alpha = \bigcap_{\alpha\in\Lambda}(X \setminus A_\alpha)$ is closed, and so $X \setminus (X \setminus \bigcup_{\alpha\in\Lambda} A_\alpha) = \bigcup_{\alpha\in\Lambda} A_\alpha$ is open.

Remark 3.1.11. The intersection of two open sets is not an open set in general as can be seen from the following example.

Example 3.1.12. From Example 3.1.8, we obtain that $\{1,3\}$ and $\{2,3\}$ are open but $\{1,3\} \cap \{2,3\} = \{3\}$ is not open.

3.2 T_0-bi-w spaces and T_1-bi-w spaces

In this section, we will introduce the notions of T_0-bi-w spaces, T_1-bi-w spaces and investigate some of their properties.

Definition 3.2.1. A bi-w space (X, w^1, w^2) is called a T_0-*bi-w space* if for any pair of distinct points x and y in X, there exists a subset U which is either w^1-open or w^2-open such that $x \in U, y \notin U$ or $y \in U, x \notin U$.

Example 3.2.2. Let $X = \{1, 2, 3\}$. Define WS w^1 and w^2 on X as follows:

$$w^1 = \{\emptyset, \{1\}, \{2, 3\}\} \text{ and } w^2 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

Consider $x, y \in X$ such that $x \neq y$. We have the following six cases:

case $x = 1, y = 2$, there exists w^1-open set $U = \{1\}$ such that $1 \in \{1\}, 2 \notin \{1\}$;

case $x = 2, y = 1$, there exists w^2-open set $U = \{2\}$ such that $2 \in \{2\}, 1 \notin \{2\}$;

case $x = 1, y = 3$, there exists w^1-open set $U = \{1\}$ such that $1 \in \{1\}, 3 \notin \{1\}$;

case $x = 3, y = 1$, there exists w^1-open set $U = \{2, 3\}$ such that $3 \in \{2, 3\}, 1 \notin \{2, 3\}$;

case $x = 2, y = 3$, there exists w^2-open set $U = \{2\}$ such that $2 \in \{2\}, 3 \notin \{2\}$;

case $x = 3, y = 2$, there exists w^2-open set $U = \{1, 2\}$ such that $2 \in \{1, 2\}, 3 \notin \{1, 2\}$.

Thus, X is a T_0-bi-w space.

Now, we give a characterization of T_0-bi-w space.

Theorem 3.2.3. A bi-w space (X, w^1, w^2) is a T_0-bi-w space if and only if for any pair of distinct points x and y in X, $c_{w^1}(\{x\}) \neq c_{w^1}(\{y\})$ or $c_{w^2}(\{x\}) \neq c_{w^2}(\{y\})$.

Proof. Suppose that X is a T_0-bi-w space and let $x, y \in X$ such that $x \neq y$. Then there exists a subset U which is either w^1-open or w^2-open such that $x \in U, y \notin U$ or $y \in U, x \notin U$. If $x \in U, y \notin U$ and U is w^i-open where $i \in \{1, 2\}$, then $U \cap \{y\} = \emptyset$. Thus, $x \notin c_{w^i}(\{y\})$, but $x \in c_{w^i}(\{x\})$. Hence, $c_{w^i}(\{x\}) \neq c_{w^i}(\{y\})$. For the case $y \in U, x \notin U$, the proof is similar.

Conversely, suppose that $x, y \in X$ such that $x \neq y$. If $c_{w^1}(\{x\}) \neq c_{w^1}(\{y\})$, then there exists a point $z \in X$ such that

$$z \in c_{w^1}(\{x\}) \text{ and } z \notin c_{w^1}(\{y\}),$$

or

$$z \in c_{w^1}(\{y\}) \text{ and } z \notin c_{w^1}(\{x\}).$$

Without loss generality, we can assume that $z \in c_{w^1}(\{x\})$ and $z \notin c_{w^1}(\{y\})$. Thus, there exists a w^1-open subset U such that $z \in U$ and $U \cap \{y\} = \emptyset$, that is $y \notin U$. Since $z \in c_{w^1}(\{x\})$, $U \cap \{x\} \neq \emptyset$. Thus $x \in U$. Similarly, if $c_{w^2}(\{x\}) \neq c_{w^2}(\{y\})$, then there exists a w^2-open subset U such that $y \in U, x \notin U$. Thus X is a T_0-bi-w space.

Definition 3.2.4. A bi-w space (X, w^1, w^2) is called a T_1-*bi-w space* if for any pair of distinct points x and y in X, there exist a w^1-open set U and a w^2-open set V such that $x \in U, y \notin U$ and $y \in V, x \notin V$.

Example 3.2.5. Let $X = \{1, 2, 3\}$. Define WS w^1 and w^2 on X as follows:

$$w^1 = \{\emptyset, \{1,2\}, \{1,3\}, \{2,3\}\} \text{ and } w^2 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{2,3\}\}.$$

Consider x, $y \in X$ such that $x \neq y$. We have the following six cases:

case $x = 1, y = 2$, there exist w^1-open set $U = \{1, 3\}$ and a w^2-open set $V = \{2, 3\}$ such that $1 \in \{1, 3\}$, $2 \notin \{1, 3\}$ and $2 \in \{2, 3\}$, $1 \notin \{2, 3\}$;

case $x = 2, y = 1$, there exist w^1-open set $U = \{2, 3\}$ and a w^2-open set $V = \{1\}$ such that $2 \in \{2, 3\}$, $1 \notin \{2, 3\}$ and $1 \in \{1\}$, $2 \notin \{1\}$;

case $x = 1, y = 3$, there exist w^1-open set $U = \{1, 2\}$ and a w^2-open set $V = \{2, 3\}$ such that $1 \in \{1, 2\}$, $3 \notin \{1, 2\}$ and $3 \in \{2, 3\}$, $1 \notin \{2, 3\}$;

case $x = 3, y = 1$, there exist w^1-open set $U = \{2, 3\}$ and a w^2-open set $V = \{1\}$ such that $3 \in \{2, 3\}$, $1 \notin \{2, 3\}$ and $1 \in \{1\}$, $3 \notin \{1\}$;

case $x = 2, y = 3$, there exist w^1-open set $U = \{1, 2\}$ and a w^2-open set $V = \{3\}$ such that $2 \in \{1, 2\}$, $3 \notin \{1, 2\}$ and $3 \in \{3\}$, $2 \notin \{3\}$;

case $x = 3, y = 2$, there exist w^1-open set $U = \{1, 3\}$ and a w^2-open set $V = \{2\}$ such that $3 \in \{1, 3\}$, $2 \notin \{1, 3\}$ and $2 \in \{2\}$, $3 \notin \{2\}$.

Thus, X is a T_1-bi-w space.

Remark 3.2.6. From Definition 3.2.4, if (X, w^1, w^2) is a T_1-bi-w space and $x, y \in X$ such that $y \neq x$, then there exist a w^1-open set U containing y but not x and a w^2-open set Vcontaining y but not x.

Theorem 3.2.7. Let (X, w^1, w^2) be a bi-w space. Then X is a T_1-bi-w space if and only if every singleton subset of X is closed.

Proof . Assume that X is a T_1-bi-w space and let $x \in X$. We will show that $\{x\} = c_{w^1}(\{x\})$ and $\{x\} = c_{w^2}(\{x\})$. Let $y \in X$ such that $y \neq x$. By Lemma 3.2.6, there exist a w^1-open set U and a w^2-open set V such that $y \in U, x \notin U$ and $y \in V, x \notin V$. Thus $U \cap \{x\} = \emptyset$ and $V \cap \{x\} = \emptyset$. This implies that $y \notin c_{w^1}(\{x\})$ and $y \notin c_{w^2}(\{x\})$. Hence, $\{x\} = c_{w^1}(\{x\})$ and $\{x\} = c_{w^2}(\{x\})$, and so $\{x\}$ is closed in X.

Conversely, assume that every singleton subset of X is closed. Let $x, y \in X$ such that $x \neq y$. By assumption, we have $\{x\} = c_{w^2}(\{x\})$ and $\{y\} = c_{w^1}(\{y\})$. Since $x \notin c_{w^1}(\{y\})$ and $y \notin c_{w^2}(\{x\})$, there exist a w^1-open set U and a w^2-open set V such that $x \in U, U \cap \{y\} = \emptyset$ and $y \in V, V \cap \{x\} = \emptyset$. Then $x \in U, y \notin U$ and $\in V, x \notin V$. Hence, X is a T_1-bi-w space.

Lemma 3.2.8. Let (X, w^1, w^2) be a bi-w space. If X is a T_1-bi-w space, then X is a T_0-bi-w space.

Proof. It follows from Definition 3.2.1 and Definition 3.2.4.

Remark 3.2.9. We can see from the following example that the converse of the above lemma is not true.

Example 3.2.10. Let $X = \{1, 2, 3\}$. Define WS w^1 and w^2 on X as follows:

$$w^1 = \{\emptyset, \{1\}, \{3\}, \{1, 3\}\} \text{ and } w^2 = \{\emptyset, \{1\}, \{3\}, \{1, 2\}, \{1, 3\}, X\}.$$

Then X is a T_0-bi-w space. But it is not a T_1-bi-w space.

Definition 3.2.11. Let $i, j \in \{1, 2\}$ where $i \neq j$. A bi-w space (X, w^1, w^2) is said to be a $R_0^{(i,j)}$-*bi-w space* if for every w^i-open set U and for each $x \in U, c_{w^i}(c_{w^j}(\{x\})) \subset U$. A bi-w space (X, w^1, w^2) is called a R_0-*bi-w space* if X is a $R_0^{(i,j)}$-bi-w space and a $R_0^{(j,i)}$-bi-w space.

Example 3.2.12. Let $X = \{1, 2, 3\}$. Define WS w^1 and w^2 on X as follows:

$$w^1 = \{\emptyset, \{1\}, \{2, 3\}\} \text{ and } w^2 = \{\emptyset, \{1\}, \{2, 3\}, X\}.$$

Since $c_{w^1}\big(c_{w^2}(\{1\})\big) = \{1\} = c_{w^2}\big(c_{w^1}(\{1\})\big)$, $c_{w^1}\big(c_{w^2}(\{2\})\big) = \{2, 3\} = c_{w^2}\big(c_{w^1}(\{2\})\big)$ and $c_{w^1}\big(c_{w^2}(\{3\})\big) = \{2, 3\} = c_{w^2}\big(c_{w^1}(\{3\})\big)$, X is a $R_0^{(1,2)}$-bi-w space and a $R_0^{(2,1)}$-bi-w space. This implies that X is a R_0-bi-w space.

Theorem 3.2.13. Let (X, w^1, w^2) be a bi-w space. Then the following are equivalent:

1. X is a R_0-bi-w space;
2. For each $x, y \in X$, if $x \notin c_{w^1}(\{y\})$, then $y \notin c_{w^1}(c_{w^2}(\{x\}))$ and if $x \notin c_{w^2}(\{y\})$, then $y \notin c_{w^2}(c_{w^1}(\{x\}))$;
3. For each $x, y \in X$, if $x \in c_{w^1}(c_{w^2}(\{y\}))$, then $y \in c_{w^1}(\{x\})$ and if $x \in c_{w^2}(c_{w^1}(\{y\}))$, then $y \in c_{w^2}(\{x\})$.

*Proof.*1. $\Longrightarrow$ 2. Suppose that X is R_0-bi-w space. Let $x, y \in X$ with $x \notin c_{w^1}(\{y\})$. So there exists a w^1-open subset U such that $x \in U \subset X \setminus \{y\}$. Since X is a R_0-bi-w space, then $c_{w^1}(c_{w^2}(\{x\})) \subset U$. It follows that $y \notin c_{w^1}(c_{w^2}(\{x\}))$. Similarly, if $x \notin c_{w^2}(\{y\})$, than $y \notin c_{w^2}(c_{w^1}(\{x\}))$.

2. $\Longrightarrow$ 3. It is obvious.

3. $\Longrightarrow$ 1. Let $i,j \in \{1,2\}$ where $i \neq j$. Let U be a w^i-open set and let $x \in U$. Assume that $y \in c_{w^i}(c_{w^j}(\{x\}))$. By assumption, we have $x \in c_{w^i}(\{y\})$. Thus $U \cap \{y\} \neq \emptyset$, and hence $y \in U$. This implies that $c_{w^i}(c_{w^j}(\{x\})) \subset U$. Thus X is a R_0-bi-w space.

Proposition 3.2.14. Let (X, w^1, w^2) be a R_0-bi-w space. Then $x, y \in X$, $c_{w^i}(\{x\}) = c_{w^i}(\{y\})$ or $c_{w^i}(\{x\}) \cap c_{w^i}(\{y\}) = \emptyset$ where $i \in \{1,2\}$.

Proof. Let $i \in \{1,2\}$. Suppose that (X, w^1, w^2) is a R_0-bi-w space and $x, y \in X$. If $c_{w^i}(\{y\}) \cap c_{w^i}(\{x\}) \neq \emptyset$, then there exists $z \in c_{w^i}(\{x\}) \cap c_{w^i}(\{y\})$. So $z \in c_{w^i}(\{x\}) \subset c_{w^i}(c_{w^j}(\{x\}))$ where $j \in \{1,2\}$ and $i \neq j$. By Theorem 3.2.13(3), $x \in c_{w^i}(\{z\})$. It follows that $c_{w^i}(\{z\}) \subset c_{w^i}(\{x\})$ and $c_{w^i}(\{x\}) \subset c_{w^i}(\{z\})$. Consequently, $c_{w^i}(\{z\}) = c_{w^i}(\{x\})$. In the same way, $c_{w^i}(\{z\}) = c_{w^i}(\{y\})$. So $c_{w^i}(\{x\}) = c_{w^i}(\{y\})$.

Lemma 3.2.15. Let (X, w^1, w^2) be a R_0-bi-w space and $x, y \in X$, if $x \in c_{w^i}(c_{w^j}(\{y\}))$, then $y \in c_{w^j}(c_{w^i}(\{x\}))$ where $i, j \in \{1,2\}$ and $i \neq j$.

Proof. Suppose that $x \in c_{w^i}(c_{w^j}(\{y\}))$. By Theorem 3.2.13(3), $y \in c_{w^i}(\{x\})$. Thus $c_{w^j}(\{y\}) \subset c_{w^j}(c_{w^i}(\{x\}))$. Therefore, $y \in c_{w^j}(c_{w^i}(\{x\}))$.

Lemma 3.2.16. Let (X, w^1, w^2) be a bi-w space. If X is a T_1-bi-w space, then X is a R_0-bi-w space.

Proof. Suppose that X is a T_1-bi-w space. Let U be a w^1-open set, and let $x \in U$. Since X is a T_1-bi-w space, and by Theorem 3.2.7, $c_{w^1}\left(c_{w^2}(\{x\})\right) = \{x\} \subset U$. Then X is a $R_0^{(1,2)}$-bi-w space. Similarly, we can prove that X is a $R_0^{(2,1)}$-bi-w space. Therefore (X, w^1, w^2) is a R_0-bi-w space.

Theorem 3.2.17. Let (X, w^1, w^2) be a bi-w space. Then the following are equivalent:

1. X is a T_1-bi-w space;
2. X is a T_0-bi-w space and a R_0-bi-w space.

Proof. 1. $\Longrightarrow$ 2. It follows from Lemma 3.2.8 and Lemma 3.2.16.

2. $\Longrightarrow$ 1. Suppose that X is a T_0-bi-w space and a R_0-bi-w space. We will show that X is a T_1-bi-w space, i.e., $\{x\}$ is closed for all $x \in X$. Let $x \in X$ and let $y \in X$ such that $y \neq x$. Since X is a T_0-bi-w space, there exists a w^1-open set or a w^2-open set U such that $x \in U, y \notin U$ or $y \in U, x \notin U$. Without loss of generality, we assume that U is a w^1-open set such that $x \in U, y \notin U$. Since X is a R_0-bi-w space, $c_{w^1}(c_{w^2}(\{x\})) \subset U$, thus $y \notin c_{w^1}(c_{w^2}(\{x\}))$. Since $c_{w^1}(\{x\}) \subset c_{w^1}(c_{w^2}(\{x\}))$, $y \notin c_{w^1}(\{x\})$. Similarly, we can prove that $y \notin c_{w^2}(\{x\})$. This implies that $c_{w^1}(\{x\}) \subset \{x\}$ and $c_{w^2}(\{x\}) \subset \{x\}$. Thus $\{x\}$ is closed. Hence X is a T_1-bi-w space.

Conclusions

In this paper, we introduced the concept on a bi-weak structure space or briefly a bi-w space. We also studied closed sets and some separation axioms in this space. We obtained some characterizations of closed sets. In particular, a closed set in a bi-w space can be written in the from of its closure in the weak structures. Moreover, we showed that an arbitrary intersection of closed sets is closed. We also gave some characterizations of a T_0-bi-w space, a T_1-bi-w space and a R_0-bi-w space. In particular, a T_1-bi-w space is both a T_0-bi-w space and a R_0-bi-w space. In this paper, we find several properties as of the space presented; however, there are many interesting properties to be further investigated.

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