ทฤษฎีบทการลู่เข้าอย่างเข้มสำหรับปัญหาสมดุลผสมและการส่งหลายค่าแบบไม่ขยายกึ่งเชิงเส้นกำกับทุกส่วนเบรกแมนเอกรูปในปริภูมิบานาคสะท้อน

Strong Convergence Theorems for Mixed Equilibrium Problems and Uniformly Bregman Totally Quasi-Asymptotically Nonexpansive Multi-Valued Mappings in Reflexive Banach Spaces

กิตติศักดิ์ จันทะการ และ อัญชลีย์ แก้วเจริญ*

Kittisak Jantakarn and Anchalee Kaewcharoen*

ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยนเรศวร

Department of Mathematics, Faculty of Science, Naresuan University

Received : 24 October 2017

Accepted : 5 February 2018

Published online : 9 February 2018

บทคัดย่อ

ในบทความวิจัยนี้ ผู้จัดทำได้เสนอกระบวนการทำซ้ำสำหรับหาคำตอบร่วมของปัญหาสมดุลผสมและปัญหาจุดตรึงของการส่งหลายค่าแบบไม่ขยายกึ่งเชิงเส้นกำกับทุกส่วนเบรกแมนเอกรูปในปริภูมิบานาคสะท้อน และพิสูจน์ทฤษฎีบทการลู่เข้าอย่างเข้มภายใต้เงื่อนไขควบคุมที่เหมาะสมบางอย่าง

คำสำคัญ : ปัญหาสมดุลผสม การส่งหลายค่าแบบไม่ขยายกึ่งเชิงเส้นกำกับทุกส่วนเบรกแมน ปริภูมิบานาคสะท้อน

Abstract

In this paper, we propose a new iterative method for finding common solutions of mixed equilibrium problems and common fixed points of uniformly Bregman totally quasi-asymptotically nonexpansive multi-valued mappings in reflexive Banach spaces and prove the strong convergence theorems under some suitable control conditions.

Keywords : mixed equilibrium problems, Bregman totally quasi-asymptotically nonexpansive multi-valued mappings, reflexive Banach spaces.

***Corresponding author. E-mail** : anchaleeka@nu.ac.th

Introduction

Let R be the set of all real numbers, E be a real reflexive Banach space with the dual E^* and C be a nonempty closed convex subset of E. Let $G : C \times C \to R$ be a bifunction and $\varphi : C \to R$ be a real-valued function. In 1994, Blum and Oettli (Blum & Oettli, 1994) firstly studied the equilibrium problem:

$$(EP) \qquad \text{finding } x \in C \text{ such that } G(x, y) \geq 0 \text{ for all } y \in C .$$

In 2008, the equilibrium problem was generalized by Ceng and Yao (Ceng & Yao, 2008) to the mixed equilibrium problem:

$$(MEP) \qquad \text{finding } x \in C \text{ such that } G(x, y) + \varphi(y) - \varphi(x) \geq 0 \text{ for all } y \in C .$$

The mixed equilibrium problems include fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems and the equilibrium problems.

It turns out that the fixed point theory of nonexpansive mappings can be applied to solving solutions of certain evolution equations and solving convex feasibility, variational inequality and equilibrium problems. There are, in fact, many papers that deal with methods for finding fixed points of nonexpansive and quasi-nonexpansive mappings in Hilbert, uniformly convex and uniformly smooth Banach spaces.

In 1990, Kirk and Massa (Kirk & Massa, 1990) generalized the fixed point theorems for single-valued nonexpansive mappings to multi-valued nonexpansive mappings and proved the existence of fixed points in Banach spaces. Thereafter, many researchers generalized multi-valued nonexpansive mappings and obtained fixed point theorems under some suitable control conditions.

Let $N(C)$ and $CB(C)$ denote the families of nonempty subsets and nonempty closed bounded subsets of C, respectively. The Hausdorff metric on $CB(C)$ is defined by

$$H(A, B) = \max\left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\},$$

for all $A, B \in CB(C)$ where $d(x, B) = \inf\{\| x - y \|, y \in B\}$.

In 1967, Bregman (Bregman, 1967) discovered a technique using the Bregman distance function $D_f(\cdot,\cdot)$ in designing and analyzing optimization and feasibility algorithms. Bregman's technique has been applied in various ways.

When we try to extend the results to general Banach spaces we encounter some difficulties and there are several ways to overcome these difficulties. One of them is to use the Bregman distances instead of the norm, Bregman (quasi-) nonexpansive mappings instead of the (quasi-) nonexpansive mappings and the Bregman projections instead of the metric projections.

In 2013, Li et al. (Li *et al.*, 2013) introduced the concept of Bregman strongly nonexpansive multi-valued mappings and obtained strong convergence theorems for the modify Halpern's iterations. Moreover, the application for solving equilibrium problems in the framework of reflexive Banach spaces is presented.

In 2014, Li and Liu (Li & Liu, 2014) introduced the concept of Bregman totally quasi-asymptotically nonexpansive multi-valued mappings and proved strong convergence theorems for the hybrid Halpern's iterations for a countable family of Bregman totally quasi-asymptotically nonexpansive multi-valued mappings. Moreover, they (Li & Liu, 2014) applied their main results to solve classical equilibrium problems in the framework of reflexive Banach spaces.

In this paper, we propose a new iterative method for finding common solutions of mixed equilibrium problems and common fixed points of uniformly Bregman totally quasi-asymptotically nonexpansive multi-valued mappings in the framework of reflexive Banach spaces and prove the strong convergence theorems under some suitable control conditions.

Preliminaries

We begin by recalling some basic definitions and lemmas which will be used in our proofs.

Let E be a real reflexive Banach space, E^* be the dual space of E and $f : E \to (-\infty, +\infty]$ be a proper lower semi-continuous and convex function. We denote by $domf$ the domain of f, that is, the set $\{x \in E : f(x) < +\infty\}$.

Let $x \in int(domf)$. The subdifferential of f at x is the convex set defined by

$$\partial f(x) = \left\{ x^* \in E^* : f(x) + \langle x^*, y - x\rangle \le f(y), \forall y \in E \right\}.$$

Let $f^* : E^* \to (-\infty, +\infty]$ be the Fenchel conjugate of f defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}, \forall x^* \in E^*.$$

We know that the Young-Fenchel inequality holds, that is,

$$\langle x^*, x\rangle \le f(x) + f^*(x^*), \quad \forall x \in E, x^* \in E^*.$$

For any $x \in int(domf)$ and $y \in E$, we define the right-hand derivative of f at x the direction y by

$$f^0(x, y) = \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}.$$

In this case, $f^0(x, y)$ coincides with $\nabla f(x)$, the value of the gradient ∇f of f at x.

Definition 1 Let $f : E \to (-\infty, +\infty]$. The function f is called to be:

(1) Gateaux differentiable at x if $\lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}$ exists for any y;

(2) Gateaux differentiable if it is Gateaux differentiable for any $x \in int(domf)$;

(3) Frechet differentiable at x if the above limit is attained uniformly in $\| y\| = 1$;

(4) uniformly Frechet differentiable on a subset C of E if the above limit is attained uniformly for all $x \in C$ and $\| y\| = 1$.

Lemma 2 (Reich & Sabach, 2010) Let $f : E \to (-\infty, +\infty]$ be uniformly Frechet differentiable and bounded on bounded subsets of E. Then f is uniformly continuous on bounded subsets of E and ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.

Definition 3 A function $f : E \to (-\infty, +\infty]$ is said to be a Legendre function if the following conditions are satisfied:

(L1) The interior of the domain of f, $int(domf)$ is nonempty, f is Gateaux differentiable on $int(domf)$ and $dom\nabla f = int(domf)$;

(L2) The interior of the domain of f^*, $int(domf^*)$ is nonempty, f^* is Gateaux differentiable on $int(domf^*)$ and $dom\nabla f^* = int(domf^*)$.

Remark 4 If E is a real reflexive Banach space and f is the Legendre function, then the following conditions hold:

(a) f is the Legendre function if and only if f is the Legendre function;

(b) $(\partial f)^{-1} = \partial f^*$;

(c) $\nabla f = (\nabla f^*)^{-1}, ran\nabla f = dom\nabla f^* = int(domf^*), ran\nabla f^* = dom\nabla f = int(domf)$;

(d) the function f and f^* are strictly convex on the interior of respective domains.

Example 5 If E is a smooth and strictly convex Banach space and $f : E \to (-\infty, +\infty]$ is a function defined by $f(x) = \frac{1}{p}\|x\|^p$ $(1 < p < +\infty)$, then f is a Legendre function. Moreover, the gradient ∇f of f coincides with the generalized duality mapping of E, i.e., $\nabla f = J_p$ $(1 < p < +\infty)$. In particular, $\nabla f = I$, the identity mapping in Hilbert spaces.

From now on, we assume that the function f is Legendre.

Definition 6 (Censor & Lent, 1981) Let $f : E \to (-\infty, +\infty]$ be a convex and Gateaux differentiable function. The function $D_f : domf \times int(domf) \to [0, +\infty)$ defined by

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle$$

is called to be the Bregman distance with respect to f.

It should be noted that the Bregman distance is not a distance in the usual sense of the term. In general, $D_f(\cdot,\cdot)$ is not symmetric and does not satisfy the triangle inequality. But by the definition, we know it has the following important properties:

(1) (the two point identity) for any $x, y \in int(domf)$,

$$D_f(x, y) + D_f(y, x) = \langle \nabla f(x) - \nabla f(y), x - y \rangle ;$$

(2) (the three point identity) for any $x \in domf$ and $y, z \in int(domf)$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle ;$$

(3) (the four point identity) for any $y, w \in domf$ and $x, z \in int(domf)$,

$$D_f(y,x) - D_f(y,z) - D_f(w,x) + D_f(w,z) = \langle \nabla f(z) - \nabla f(x), y - w \rangle .$$

Definition 7 (Bregman, 1967) Let $f : E \to (-\infty, +\infty]$ be a convex and Gateaux differentiable function. The Bregman projection of x in $int(domf)$ onto the nonempty closed convex set $C \subset domf$ is the necessarily unique vector $proj_C^f(x) \in C$ satisfying the following:

$$D_f(proj_C^f(x), x) = \inf\{D_f(y,x) : y \in C\} .$$

Definition 8 (Butnariu & Iusem, 2000) Let $f : E \to (-\infty, +\infty]$ be a convex and Gateaux differentiable function. The modulus of total convexity at $x \in domf$ is the function $v_f : int(domf) \times [0, +\infty) \to [0, +\infty)$ defined by

$$v_f(x,t) := \inf\{D_f(y,x) : y \in domf, \| y - x \| = t\} .$$

A function f is called to be

(a) totally convex at a point $x \in int(domf)$ if $v_f(x,t)$ is positive whenever $t > 0$;

(b) totally convex if it is totally convex at every point $x \in int(domf)$;

(c) totally convex on bounded sets if $v_f(B,t)$ is positive for any nonempty bounded subset B of E and $t > 0$ where the modulus of total convexity of the function f on the set B is the function $v_f : int(domf) \times [0, +\infty) \to [0, +\infty)$ defined by

$$v_f(B,t) := \inf\{v_f(x,t) : x \in B \cap domf\} .$$

Lemma 9 (Reich & Sabach, 2010) If $x \in int(domf)$, then the following statements are equivalent:

(1) the function f is totally convex at x ;

(2) for any sequence $\{y_n\} \subset \mathrm{dom} f, \lim_{n\to\infty} D_f(y_n, x) = 0 \Rightarrow \lim_{n\to\infty} \| y_n - x \| = 0$.

Lemma 10 (Butnariu & Iusem, 2000) The function f is totally convex on bounded sets if and only if it is sequentially consistent, i.e., for any two sequences and $\{x_n\}$ and $\{y_n\}$ in $int(domf)$ and $domf$, respectively, and $\{x_n\}$ is bounded, then

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} \| y_n - x_n \| = 0 .$$

Lemma 11 (Reich & Sabach, 2010) Let $f : E \to (-\infty, +\infty]$ be a Gateaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.

Lemma 12 (Butnariu & Resmerita, 2006) Let $f : E \to (-\infty, +\infty]$ be a Gateaux differentiable function and totally convex on $int(domf)$. Let $x \in int(domf)$ and $C \subset int(domf)$ be a nonempty closed convex set. If $x \in C$, then the following statements are equivalent

(1) $z \in C$ is the Bregman projection of x onto C with respect to f , i.e., $z = proj_C^f(x)$;

(2) the vector z is the unique solution of the variational inequality:

$$\langle \nabla f(x) - \nabla f(z), z - y \rangle \geq 0, \forall y \in C ;$$

(3) the vector z is the unique solution of the inequality:

$$D_f(y,z) + D_f(z,x) \leq D_f(y,x), \forall y \in C .$$

Definition 13 (Li & Liu, 2014) Let C be a convex subset of $int(domf)$ and $T: C \to N(C)$ be a multi-valued mapping. A point $p \in C$ is called to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T and $F(T)$ the set of fixed points of T. That is $F(T) = \{x \in C : x \in Tx\}$.

Definition 14 Let C be a subset of E and $T: C \to N(C)$ be a multi-valued mapping with a nonempty fixed point set. A mapping T is called to be:

(a) Bregman firmly nonexpansive if

$$D_f(x^*, y^*) + D_f(y^*, x^*) + D_f(x^*, x) + D_f(y^*, y) \le D_f(x^*, y) + D_f(y^*, x), \quad \forall x, y \in C, x^* \in Tx, y^* \in Ty;$$

(b) Bregman strongly nonexpansive with respect to a nonempty $\hat{F}(T)$ if

$$D_f(p, z) \le D_f(p, x), \forall x \in C, p \in \hat{F}(T), z \in Tx$$

and if whenever $\{x_n\} \subset C$ is bounded, $p \in \hat{F}(T)$ and

$$\lim_{n\to\infty}[D_f(p, x_n) - D_f(p, z_n)] = 0,$$

then $\lim_{n\to\infty} D_f(x_n, z_n) = 0$ where $z_n \in Tx_n$;

(c) Bregman relatively nonexpansive if $F(T) = \hat{F}(T)$ and

$$D_f(p, z) \le D_f(p, x), \forall x \in C, p \in F(T), z \in Tx;$$

(d) Bregman quasi-nonexpansive if

$$D_f(p, z) \le D_f(p, x), \forall x \in C, p \in F(T), z \in Tx;$$

(e) Bregman quasi-asymptotically nonexpansive if there exists a real sequence $\{k_n\} \subset [1, +\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$D_f(p, z) \le k_n D_f(p, x), \forall x \in C, p \in F(T), z \in T^n x; \quad (1)$$

(f) Bregman totally quasi-asymptotically nonexpansive, if there exist nonnegative real sequences $\{v_n\}, \{\mu_n\}$ with $v_n \to 0, \mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta: R^+ \to R^+$ with $\zeta(0) = 0$ that

$$D_f(p, z) \le D_f(p, x) + v_n \zeta(D_f(p, x)) + \mu_n, \forall n \ge 1, \forall x \in C, p \in F(T), z \in T^n x;$$

(g) closed if for any sequence $\{C_n\}$ where $C_n \subset C$ for all $n \ge 1$ with $x \in C$ and $d(T(C_n), y) \to 0$ where $y \in C$, then $y \in Tx$.

Remark 15 From these definitions, it is obvious that

(1) each Bregman relatively nonexpansive mapping is a Bregman quasi-nonexpansive mapping;

(2) each Bregman quasi-nonexpansive mapping is a Bregman quasi-asymptotically nonexpansive mapping. If taking, $k_n = 1$, then we have

$$D_f(p,z) \le D_f(p,x) = k_n D_f(p,x), \forall x \in C, p \in F(T), z \in T^n x ;$$

(3) each Bregman quasi-asymptotically nonexpansive mapping is a Bregman totally quasi-asymptotically nonexpansive mapping. If taking, $\zeta(t) = t, t \ge 0, v_n = k_n - 1$ and $\mu_n = 0$, then equation (1) can be rewritten as

$$D_f(p,z) \le D_f(p,x) + v_n \zeta(D_f(p,x)) + \mu_n, \forall x \in C, p \in F(T), z \in T^n x .$$

This implies that each Bregman relatively nonexpansive mapping must be a Bregman totally quasi-asymptotically nonexpansive mapping but the converse is not true.

Let C be a nonempty closed convex subset of a real reflexive Banach space E , and let $G : C \times C \to R$ be a bifunction satisfying the following conditions:

(C1) $G(x,x) = 0, \forall x \in C$;

(C2) $G(x,y) + G(y,x) \le 0, \forall x, y \in C$ (G is monotone);

(C3) $\limsup_{t \to 0^+} G(tz + (1-t)x, y) \le G(x,y), \forall x, y, z \in C$;

(C4) $G(x,\cdot)$ is convex and lower semi-continuous, $\forall x \in C$.

Definition 16 A function $f : E \to (-\infty, +\infty]$ is said to be

(1) coercive if $\lim_{\|x\| \to +\infty} f(x) = +\infty$;

(2) strong coercive if $\lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty$.

Lemma 17 (Darvish, 2015) Let $f : E \to (-\infty, +\infty]$ be a strong coercive Legendre function and C be a nonempty closed convex subset of $int(domf)$. Let $\varphi : C \to R$ be a proper lower semi-continuous and convex function. Assume that $G : C \times C \to R$ satisfies conditions (C1)-(C4). For $x \in E$, define a mapping $Res_{G,\varphi}^f : E \to 2^C$ as follows:

$$Res_{G,\varphi}^f(x) = \{z \in C : G(z,y) + \varphi(y) - \varphi(z) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \ge 0, \forall y \in C\} .$$

Then the following results hold:

(1) $Res_{G,\varphi}^f$ is single-valued and $dom(Res_{G,\varphi}^f) = E$;

(2) $Res_{G,\varphi}^f$ is Bregman firmly nonexpansive;

(3) $MEP(G,\varphi)$ is a closed and convex subset of C and $MEP(G,\varphi) = F(Res_{G,\varphi}^f)$;

(4) for all $x \in E, u \in F(Res_{G,\varphi}^f)$, $D_f(u, Res_{G,\varphi}^f x) + D_f(Res_{G,\varphi}^f x, x) \le D_f(u,x)$.

Lemma 18 (Kassay *et al.*, 2011) Let $f : E \to (-\infty, +\infty]$ be a Legendre function such that ∇f^* is bounded on bounded subsets of $int(domf)$ and $x \in E$. If $\{D_f(x, x_n)\}$ is bounded, then the sequence $\{x_n\}$ is bounded.

Lemma 19 (Li & Liu, 2014) Let E be a real reflexive Banach space and C be a nonempty closed convex subset of E and $f : E \to (-\infty, +\infty]$ be a Legendre function. Let $T : C \to C$ be a Bregman totally quasi-asymptotically nonexpansive multi-valued mapping with respect to f . Then the fixed point set $F(T)$ of T is a closed convex subset of C .

Lemma 20 (Reich & Sabach, 2010) Let $f: E \to (-\infty, +\infty]$ be a Gateaux differentiable and totally convex function, $x_0 \in E$ and C be a nonempty closed convex subset of E . Suppose that the sequence $\{x_n\}$ is bounded and any weak subsequential limit of $\{x_n\}$ belongs to C . If $D_f(x_n, x_0) \le D_f(proj_C^f(x_0), x_0)$ for any $n \ge 1$, then $\{x_n\}$ converges strongly to $proj_C^f(x_0)$.

Definition 21

(1) A countable family of multi-valued mappings $\{T_i : C \to N(C)\}_{i=1}^{\infty}$ is said to be uniformly Bregman totally quasi-asymptotically nonexpansive if $\mathrm{F} = \bigcap_{i=1}^{\infty} F(T_i) \neq \varnothing$ and there exist nonnegative real sequences $\{v_n\}, \{\mu_n\}, v_n \to 0, \mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : R^+ \to R^+$ with $\zeta(0) = 0$, such that

$$D_f(p, z_n^i) \le D_f(p, x) + v_n \zeta(D_f(p, x)) + \mu_n, \quad p \in F(T), \forall z_n^i \in T_i^n x, x \in C .$$

(2) A multi-valued mapping $T: C \to N(C)$ is said to be uniformly L - Lipschitz continuous if there exists a constant $L > 0$ such that

$$H(T^n x, T^n y) \le L\| x - y\| , \quad \forall x, y \in C .$$

Main results

In this section, we propose a new iterative method for finding common solutions of mixed equilibrium problems and common fixed points of uniformly Bregman totally quasi-asymptotically nonexpansive multi-valued mappings in the framework of reflexive Banach spaces and prove the strong convergence theorems under some suitable control conditions.

Theorem 1 Let E be a real reflexive Banach space and $f: E \to (-\infty, +\infty]$ be a strong coercive Legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subsets of E . Let C be a nonempty closed convex subset of $int(domf)$, $\{T_i : C \to N(C)\}_{i=1}^{\infty}$ be a countable family of closed and uniformly Bregman totally quasi-asymptotically nonexpansive multi-valued mappings with nonnegative real sequences $\{v_n\}, \{\mu_n\}$ and a strictly increasing continuous function $\zeta : R^+ \to R^+$ such that $v_n \to 0, \mu_n \to 0$ (as $n \to \infty$) and $\zeta(0) = 0$ and T_i be a uniformly L_i - Lipschitz continuous for each $i \ge 1$. Let $G: C \times C \to R$ be a bifunction satisfying conditions (C1)-(C4) and $\varphi : C \to R$ be a proper lower semi-continuous and convex function. Suppose that $\Omega := \bigcap_{i=1}^{\infty} F(T_i) \cap MEP(G, \varphi) \neq \varnothing$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 = x \in C, \text{chosen arbitrarily}, \\ w_n^i : G(w_n^i, y) + \varphi(y) - \varphi(w_n^i) + \langle \nabla f(w_n^i) - \nabla f(m_n^i), y - w_n^i \rangle \geq 0, \forall y \in C, m_n^i \in T_i^n x_n, i \geq 1, \\ C_n = \{z \in C : \sup_{i \geq 1} D_f(z, w_n^i) \leq D_f(z, x_n) + \xi_n\}, \\ D_n = \bigcap_{i=1}^{n} C_i, \\ x_{n+1} = proj_{D_n}^f x, \end{cases} \tag{2}$$

where $\xi_n = v_n \sup_{v \in \Omega} \zeta(D_f(v, x_n)) + \mu_n$ and $proj_{D_n}^f$ is the Bregman projection of E onto D_n.

If $\Omega := \bigcap_{i=1}^{\infty} F(T_i) \cap MEP(g, \varphi)$ is bounded, then the sequence $\{x_n\}$ converges strongly to $\overline{x} = proj_{\Omega}^f x$.

Proof We divide the proof into 8 steps as follows:

Step 1: We show that Ω is a closed convex subset of E. By Lemma 19, we obtain that $F(T_i)$ is closed and convex for each $i \geq 1$. Using Lemma 17, we get that $MEP(G, \varphi)$ is closed and convex. This implies that Ω is also closed and convex.

Step 2: We will prove that C_n is a nonempty set for all $n \geq 1$. Let $v \in \Omega$ be given. Since $Res_{G,\varphi}^f$ is a single-valued mapping and $w_n^i = Res_{G,\varphi}^f(m_n^i)$ where $m_n^i \in T_i^n x_n$, for all $i \geq 1$, it follows from Lemma 17(4) that

$$D_f(v, Res_{G,\varphi}^f(m_n^i) + D_f(Res_{G,\varphi}^f(m_n^i), m_n^i) \leq D_f(v, m_n^i). \tag{3}$$

This implies that

$$\begin{aligned} D_f(v, Res_{G,\varphi}^f(m_n^i)) &\leq D_f(v, m_n^i) - D_f(Res_{G,\varphi}^f(m_n^i), m_n^i) \\ &\leq D_f(v, m_n^i), \forall i \geq 1. \end{aligned} \tag{4}$$

Therefore

$$\begin{aligned} D_f(v, w_n^i) &= D_f(v, \mathrm{Res}_{G,\varphi}^f(m_n^i)) \\ &\leq D_f(v, m_n^i) \\ &\leq D_f(v, x_n) + v_n \zeta(D_f(v, x_n)) + \mu_n. \end{aligned} \tag{5}$$

This implies that

$$D_f(v, w_n^i) \leq D_f(v, x_n) + \xi_n, \forall i \geq 1, \tag{6}$$

where $\xi_n = v_n \sup_{v \in \Omega} \zeta(D_f(v, x_n)) + \mu_n$. It follows that

$$\sup_{i \geq 1} D_f(v, w_n^i) \leq D_f(v, x_n) + \xi_n. \tag{7}$$

This yields $v \in C_n$ for any $n \geq 1$. Hence $\Omega \subset C_n$ and then we have $\Omega \subset D_n$.

Step 3: We show that C_n is a convex for all $n \geq 1$. Let $p, q \in C_n$ and $t \in (0,1)$. Setting $u = tp + (1-t)q$. We now prove that $u \in C_n$. Since $p, q \in C_n$, we obtain that $\sup_{i\geq 1} D_f(p, w_n^i) \leq D_f(p, x_n) + \xi_n$ and $\sup_{i\geq 1} D_f(q, w_n^i) \leq D_f(q, x_n) + \xi_n$. For each $i \geq 1$, by definition of $D_f(\cdot,\cdot)$, we have

$$f(x_n) - f(w_n^i) \leq \langle \nabla f(w_n^i), p - w_n^i \rangle - \langle \nabla f(x_n), p - x_n \rangle + \xi_n \tag{8}$$

and

$$f(x_n) - f(w_n^i) \leq \langle \nabla f(w_n^i), q - w_n^i \rangle - \langle \nabla f(x_n), q - x_n \rangle + \xi_n . \tag{9}$$

Therefore

$$f(x_n) - f(w_n^i) \leq \langle \nabla f(w_n^i), tp + (1-t)q - w_n^i \rangle - \langle \nabla f(x_n), tp + (1-t)q - x_n \rangle + \xi_n . \tag{10}$$

This implies that

$$f(x_n) - f(w_n^i) \leq \langle \nabla f(w_n^i), u - w_n^i \rangle - \langle \nabla f(x_n), u - x_n \rangle + \xi_n, \forall i \geq 1 . \tag{11}$$

Therefore $u \in C_n$ and then C_n is convex. This yields D_n is also convex.

Step 4: We show that C_n is a closed set. Let $\{z_m\} \subset C_n$ and $z_m \to z$ as $m \to \infty$. Then

$$\begin{aligned} f(x_n) - f(w_n^i) &\leq \langle \nabla f(w_n^i), z_m - w_n^i \rangle - \langle \nabla f(x_n), z_m - x_n \rangle + \xi_n \\ &= \langle \nabla f(w_n^i), z_m - z + z - w_n^i \rangle - \langle \nabla f(x_n), z_m - z + z - x_n \rangle + \xi_n \\ &= \langle \nabla f(w_n^i), z_m - z \rangle + \langle \nabla f(w_n^i), z - w_n^i \rangle - \langle \nabla f(x_n), z_m - z \rangle \\ &\quad - \langle \nabla f(x_n), z - x_n \rangle + \xi_n. \end{aligned} \tag{12}$$

Taking $m \to \infty$, we get that

$$f(x_n) - f(w_n^i) \leq \langle \nabla f(w_n^i), z - w_n^i \rangle - \langle \nabla f(x_n), z - x_n \rangle + \xi_n, \forall i \geq 1 . \tag{13}$$

Therefore $z \in C_n$ and then C_n is closed. This implied that C_n is a closed and convex set for any $n \geq 1$, then so is D_n. Thus the sequence $\{x_n\}$ is well-defined.

Step 5: We prove that $\{x_n\}$ is bounded. From $x_{n+1} = proj_{D_n}^f x$, by Lemma 12(3), we have

$$\begin{aligned} D_f(x_{n+1}, x) &= D_f(proj_{D_n}^f x, x) \\ &\leq D_f(v, x) - D_f(v, proj_{D_n}^f x) \\ &\leq D_f(v, x), \end{aligned} \tag{14}$$

for all $v \in \Omega$. Hence the sequence $\{D_f(x_{n+1}, x)\}$ is bounded. By Lemma 11, we have $\{x_n\}$ is also bounded.

Step 6: We prove that $\{x_n\}$ is a Cauchy sequence. Since $x_{n+1} = proj_{D_n}^f x$ and $x_{n+2} = proj_{D_{n+1}}^f x \in D_{n+1} \subset D_n$, from Lemma 12(3), we have

$$D_f(x_{n+2}, proj_{D_n}^f x) + D_f(proj_{D_n}^f x, x) \leq D_f(x_{n+2}, x) . \tag{15}$$

It follows that

$$D_f(x_{n+2}, x_{n+1}) + D_f(x_{n+1}, x) \leq D_f(x_{n+2}, x) . \tag{16}$$

This implies that $\{D_f(x_n, x)\}$ is nondecreasing and it is also bounded. Therefore $\lim_{n\to\infty} D_f(x_n, x)$ exists. By the construction of D_n, we define $D_n = \bigcap_{i=1}^{n} C_i$ for all $n \in \mathbb{N}$. This implies that $D_{n+1} \subset D_n$ for all $n \geq 1$ and hence $\{D_n\}$ is a decreasing sequence of sets. It follows that for any $m \geq n$, we have that $D_m \subset D_n$ and $x_m = proj^f_{D_{m-1}} x \in D_{m-1} \subset D_{n-1}$. Therefore

$$\begin{aligned} D_f(x_m, x_n) &\leq D_f(x_m, proj^f_{D_{n-1}} x) \\ &\leq D_f(x_m, x) - D_f(proj^f_{D_{n-1}} x, x) \\ &= D_f(x_m, x) - D_f(x_n, x). \end{aligned} \tag{17}$$

Letting $m, n \to \infty$, we obtain that

$$D_f(x_m, x_n) \to 0 . \tag{18}$$

It follows from Lemma 10 that

$$\lim_{m,n\to\infty} \| x_m - x_n \| = 0 . \tag{19}$$

This implies that $\{x_n\}$ is a Cauchy sequence.

Step 7: We show that $\{x_n\}$ converges to a point in $\Omega := \bigcap_{i=1}^{\infty} F(T_i) \cap MEP(G, \varphi)$. Since C is complete and $\{x_n\}$ is a Cauchy sequence, without loss of generality, we assume that

$$\lim_{n\to\infty} x_n = x^* \in C . \tag{20}$$

We now prove that $x^* \in F(T_i)$ for all $i \geq 1$. Taking $m = n+1$, we obtain that $\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0$. By Lemma 10, we have

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0 . \tag{21}$$

Since $x_{n+1} = proj^f_{D_n} x \in D_n \subset C_n$, we have

$$\sup_{i\geq 1} D_f(x_{n+1}, w^i_n) \leq D_f(x_{n+1}, x_n) + \xi_n \tag{22}$$

where $\xi_n = v_n \sup_{v\in\Omega} \zeta(D_f(v, x_n)) + \mu_n$. It follows from $\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0$ and $v_n \to 0, \mu_n \to 0$ (as $n \to \infty$), we have

$$\lim_{n\to\infty} \left(\sup_{i\geq 1} D_f(x_{n+1}, w^i_n) \right) = 0 . \tag{23}$$

Moreover, since $\sup_{i\geq 1} D_f(v, w^i_n) \leq D_f(v, x_n) + \xi_n$ and f is lower semi-continuous, we get that $\{D_f(v, x_n) + \xi_n\}_{n=1}^{\infty}$ is bounded. Therefore $\{D_f(v, w^i_n)\}_{n=1}^{\infty}$ is bounded. By Lemma 18, we obtain that $\{w^i_n\}_{n=1}^{\infty}$ bounded. This yields

$$\lim_{n\to\infty} \| x_{n+1} - w^i_n \| = 0, \forall i \geq 1 . \tag{24}$$

Since

$$\| x_n - w^i_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - w^i_n \| , \tag{25}$$

we obtain that

$$\lim_{n\to\infty}\| x_n - w_n^i\| = 0, \forall\, i \geq 1 . \tag{26}$$

Since f is uniformly Frechet differentiable, it is also uniformly continuous. It follows that

$$\lim_{n\to\infty} | f(x_n) - f(w_n^i) | = 0, \forall\, i \geq 1 . \tag{27}$$

Furthermore, we have

$$\begin{aligned} D_f(v, x_n) - D_f(v, w_n^i) &= f(v) - f(x_n) - \langle \nabla f(x_n), v - x_n \rangle - [f(v) - f(w_n^i) - \langle \nabla f(w_n^i), v - w_n^i \rangle] \\ &= f(w_n^i) - f(x_n) + \langle \nabla f(w_n^i), v - w_n^i \rangle - \langle \nabla f(x_n), v - x_n \rangle \\ &= f(w_n^i) - f(x_n) + \langle \nabla f(w_n^i), x_n - w_n^i + v - x_n \rangle - \langle \nabla f(x_n), v - x_n \rangle \\ &= f(w_n^i) - f(x_n) + \langle \nabla f(w_n^i), x_n - w_n^i \rangle + \langle \nabla f(w_n^i), v - x_n \rangle - \langle \nabla f(x_n), v - x_n \rangle \\ &= f(w_n^i) - f(x_n) + \langle \nabla f(w_n^i), x_n - w_n^i \rangle + \langle \nabla f(w_n^i) - \nabla f(x_n), v - x_n \rangle. \end{aligned} \tag{28}$$

Since $\{w_n^i\}_{n=1}^{\infty}$ is bounded, we obtain that $\{\nabla f(w_n^i)\}_{n=1}^{\infty}$ is also bounded. Therefore

$$\lim_{n\to\infty} \left(D_f(v, x_n) - D_f(v, w_n^i) \right) = 0, \forall\, i \geq 1 . \tag{29}$$

Since $w_n^i = Res_{G,\varphi}^f(m_n^i)$ where $m_n^i \in T_i^n x_n$, by Lemma 17(4), we have

$$\begin{aligned} D_f(w_n^i, m_n^i) &= D_f(Res_{G,\varphi}^f(m_n^i), m_n^i) \\ &\leq D_f(v, m_n^i) - D_f(v, Res_{G,\varphi}^f(m_n^i)) \\ &\leq D_f(v, x_n) + v_n \zeta(D_f(v, x_n)) + \mu_n - D_f(v, w_n^i). \end{aligned} \tag{30}$$

Since $\{D_f(v, x_n)\}$ is bounded, we have $\{\zeta(D_f(v, x_n))\}$ is also bounded. Using the fact that $v_n \to 0, \mu_n \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} D_f(w_n^i, m_n^i) = 0, \forall\, i \geq 1 . \tag{31}$$

By Lemma 10, we have

$$\lim_{n\to\infty}\| w_n^i - m_n^i\| = 0, \forall\, i \geq 1 . \tag{32}$$

Since

$$\| x_n - m_n^i \| \leq \| x_n - w_n^i \| + \| w_n^i - m_n^i \| , \tag{33}$$

we obtain that

$$\lim_{n\to\infty}\| x_n - m_n^i\| = 0, \forall\, i \geq 1 . \tag{34}$$

This implies that

$$\lim_{n\to\infty} d(x_n, T_i^n x_n) = 0, \forall\, i \geq 1 . \tag{35}$$

Since

$$\| x^* - m_n^i \| \leq \| x^* - x_n \| + \| x_n - m_n^i \| , \tag{36}$$

it follows that

$$\lim_{n\to\infty}\| x^* - m_n^i\| = 0, \forall\, i \geq 1 . \tag{37}$$

This implies that

$$\lim_{n\to\infty} d(x^*, T_i^n x_n) = 0, \forall i \ge 1 . \tag{38}$$

Using the uniform L_i- Lipschitz continuity of $T_i : C \to N(C)$, for each $i \ge 1$, we have

$$\begin{aligned} H(T_i^{n+1}x_n, T_i^n x_n) &\le H(T_i^{n+1}x_n, T_i^{n+1}x_{n+1}) + d(T_i^{n+1}x_{n+1}, x_{n+1}) + d(x_{n+1}, x_n) + d(x_n, T_i^n x_n) \\ &\le (L_i + 1) d(x_{n+1}, x_n) + d(T_i^{n+1}x_{n+1}, x_{n+1}) + d(x_n, T_i^n x_n). \end{aligned} \tag{39}$$

We get that $\lim_{n\to\infty} H(T_i^{n+1}x_n, T_i^n x_n) = 0$. Furthermore, we have

$$d(x^*, T_i^{n+1}x_n) \le d(x^*, T_i^n x_n) + H(T_i^n x_n, T_i^{n+1}x_n) . \tag{40}$$

Therefore $d(x^*, T_i T_i^n x_n) \to 0$ as $n \to \infty$, $\forall i \ge 1$. From the closedness of T_i , it yields that $x^* \in T_i x^*$ for each $i \ge 1$. This implies that $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$. Since f is uniformly Frechet differentiable, we obtain that ∇f is uniformly continuous on bounded sets. It follows that

$$\lim_{n\to\infty} \| \nabla f(w_n^i) - \nabla f(m_n^i) \| = 0, \forall i \ge 1 . \tag{41}$$

Since $w_n^i = Res_{G,\varphi}^f (m_n^i)$ where $m_n^i \in T_i^n x_n$, we have

$$G(w_n^i, y) + \varphi(y) - \varphi(w_n^i) + \langle \nabla f(w_n^i) - \nabla f(m_n^i), y - w_n^i \rangle \ge 0, \forall y \in C . \tag{42}$$

By using (C2), we obtain that

$$\begin{aligned} \varphi(y) - \varphi(w_n^i) + \langle \nabla f(w_n^i) - \nabla f(m_n^i), y - w_n^i \rangle &\ge -G(w_n^i, y) \\ &\ge G(y, w_n^i), \forall y \in C, i \ge 1. \end{aligned} \tag{43}$$

Since $\lim_{n\to\infty} w_n^i = x^*$ and G, φ are lower semi-continuous, we have

$$\begin{aligned} \liminf_{n\to\infty} G(y, w_n^i) &\le \liminf_{n\to\infty} \left(\varphi(y) - \varphi(w_n^i) + \langle \nabla f(w_n^i) - \nabla f(m_n^i), y - w_n^i \rangle \right) \\ &\le \limsup_{n\to\infty} \left(\varphi(y) - \varphi(w_n^i) + \langle \nabla f(w_n^i) - \nabla f(m_n^i), y - w_n^i \rangle \right), \forall i \ge 1. \end{aligned} \tag{44}$$

This implies that

$$G(y, x^*) \le \varphi(y) - \varphi(x^*) . \tag{45}$$

For any $y \in C$ and $t \in (0,1)$, let $y_t = ty + (1-t)x^* \in C$, we have

$$G(y_t, x^*) + \varphi(x^*) - \varphi(y_t) \le 0 . \tag{46}$$

Therefore

$$\begin{aligned} 0 &= G(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &= G(y_t, ty + (1-t)x^*) + \varphi(ty + (1-t)x^*) - \varphi(y_t) \\ &\le tG(y_t, y) + (1-t)G(y_t, x^*) + t\varphi(y) + (1-t)\varphi(x^*) - t\varphi(y_t) - (1-t)\varphi(y_t) \\ &= t\left[G(y_t, y) + \varphi(y) - \varphi(y_t)\right] + (1-t)\left[g(y_t, x^*) + \varphi(x^*) - \varphi(y_t)\right] \\ &\le t\left[G(y_t, y) + \varphi(y) - \varphi(y_t)\right]. \end{aligned} \tag{47}$$

It follows that

$$G(y_t, y) + \varphi(y) - \varphi(y_t) \ge 0 . \tag{48}$$

From (C3), we have

$$
\begin{aligned}
0 &\le \limsup_{t\to 0^+}\left(G(y_t, y) + \varphi(y) - \varphi(y_t)\right)\\
&= \limsup_{t\to 0^+}\left(G(ty + (1-t)x^*, y) + \varphi(y) - \varphi(y_t)\right)\\
&\le G(x^*, y) + \varphi(y) - \varphi(x^*).
\end{aligned} \tag{49}
$$

This shows that $x^* \in MEP(G,\varphi)$. To sum up, we have $x^* \in \Omega := \bigcap_{i=1}^{\infty} F(T_i) \cap MEP(G,\varphi)$.

Step 8: We show that $\{x_n\}$ converges strongly to $\overline{x} = proj_{\Omega}^{f} x$. By Lemma 17 and Lemma 19, we obtain that $\bigcap_{i=1}^{\infty} F(T_i) \cap MEP(G,\varphi)$ is a nonempty closed convex subset of E . Therefore $proj_{\Omega}^{f} x$ is well-defined. Since $proj_{\Omega}^{f} x \in \Omega \subset D_n \subset D_{n-1}$, it follows from $x_n = proj_{D_{n-1}}^{f} x$ that

$$
D_f(x_n, x) \le D_f(proj_{\Omega}^{f} x, x) . \tag{50}
$$

Using Lemma 20, we have $x_n \to proj_{\Omega}^{f} x$ (as $n \to \infty$). Therefore the sequence $\{x_n\}$ converges strongly to $\overline{x} = proj_{\Omega}^{f} x$. This completes the proof.

Remark 2 In the proof of Theorem 1, we can observe that from the iteration (2), we choose $x_1 = x \in C$. For each $i \ge 1$, we choose $m_1^i \in T_i^1 x_1$. By Lemma 17, we obtain that

$$
Res_{G,\varphi}^{f}(m_1^i) = \{ z \in C : G(z,y) + \varphi(y) - \varphi(z) + \langle \nabla f(z) - \nabla f(m_1^i), y - z\rangle \ge 0, \forall y \in C \}
$$

is single-valued. Therefore we can suppose that the element in this set is w_1^i . It follows that $w_1^i \in C$ and $G(w_1^i, y) + \varphi(y) - \varphi(w_1^i) + \langle \nabla f(w_1^i) - \nabla f(m_1^i), y - w_1^i\rangle \ge 0, \forall y \in C$. In the proof of Theorem 1, we prove that C_1 is a nonempty closed convex set and then D_1 is also a nonempty closed convex set. Therefore we can find $x_2 \in C$. It follows that $\{x_n\}$ is well- defined.

Example 3 (Chang et al., 2013, Example 2.11) Let C be the unit ball in a real Hilbert space l^2 and $f(x) = \|x\|^2$. Since $\nabla f(y) = 2y$, the Bregman distance with respect to f is

$$
D_f(x, y) = \|x\|^2 - \|y\|^2 - 2 < y, x - y > = \|x - y\|^2 , \forall x, y \in C .
$$

Let $\{T_i\}_{i=1}^{\infty} : C \to N(C)$ be a family of multi-valued mappings defined by

$$
T_i(\{x_1^{(i)}, x_2^{(i)}, \ldots\}) = \{\{0, (x_1^{(i)})^2, a_2 x_2^{(i)}, a_3 x_3^{(i)}, \ldots\}\} , \ \forall \{x_1^{(i)}, x_2^{(i)}, \ldots\} \in C ,
$$

where $\{a_j^{(i)}\}_{j=1}^{\infty}$ is a sequence in $(0,1)$ such that $\prod_{j=2}^{\infty} a_j = \dfrac{1}{2}$. Let $\sqrt{k_1} = 2$ and $\sqrt{k_n} = 2\prod_{j=2}^{n} a_j , n \ge 2$. Therefore $\lim_{n\to\infty} k_n = 1$. Letting $v_n = k_n - 1 \,(n \ge 2)$, $\zeta(t) = t \,(t \ge 0)$ and $\{\mu_n\}$ be a nonnegative real sequence with $\mu_n \to 0$ (as $n \to \infty$). Chang et al. (Chang *et al*., 2013) proved that $\{T_i\}_{i=1}^{\infty}$ is a family of closed and uniformly Bregman totally quasi-asymptotically nonexpansive mulit-valued mappings with nonnegative real sequences $\{v_n\},\{\mu_n\}$ and a strictly increasing continuous function $\zeta : R^+ \to R^+$ such that $v_n \to 0, \mu_n \to 0$ (as $n \to \infty$) and $\zeta(0) = 0$ and T_i be a uniformly L_i- Lipschitz continuous for each $i \ge 1$. Let $G(x, y) = 0$ for all $x, y \in C$ and $\varphi(x) = 0$ for all $x \in C$. Let $\{x_n\}$ be a sequence generated by (2), then $\{x_n\}$ converges strongly to $\overline{x} = proj_{\Omega}^{f} x_1$ where $\Omega := \bigcap_{i=1}^{\infty} F(T_i) \cap MEP(g, \varphi)$ is nonempty.

If in Theorem 1, we consider a single-valued Bregman totally quasi-asymptotically nonexpansive mapping and setting $\varphi \equiv 0$, we obtain the following corollary.

Corollary 4 Let E be a real reflexive Banach space and $f : E \to (-\infty, +\infty]$ be a strong coercive Legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subsets of E. Let C be a nonempty closed convex subset of $int(domf)$, $T : C \to N(C)$ be a closed and Bregman totally quasi-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences $\{v_n\}, \{\mu_n\}$ and a strictly increasing continuous function $\zeta : R^+ \to R^+$ such that $v_n \to 0, \mu_n \to 0$ (as $n \to \infty$) and $\zeta(0) = 0$. Let $G : C \times C \to R$ be a bifunction satisfying conditions (C1)-(C4). Assume that T is uniformly L - Lipschitz continuous and $\Pi = F(T) \cap EP(G) \neq \varnothing$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 = x \in C, \text{chosen arbitrarily}, \\ w_n : G(w_n, y) + \langle \nabla f(w_n) - \nabla f(m_n), y - w_n \rangle \geq 0, \forall y \in C, m_n \in T^n x_n, \\ C_n = \{z \in C : D_f(z, w_n) \leq D_f(z, x_n) + \xi_n\}, \\ D_n = \bigcap_{i=1}^{n} C_i, \\ x_{n+1} = proj_{D_n}^{f} x, \end{cases} \tag{51}$$

where $\xi_n = v_n \sup_{v \in \Pi} \zeta(D_f(v, x_n)) + \mu_n$ and $proj_{D_n}^{f}$ is the Bregman projection of E onto D_n. If Π is bounded, then the sequence $\{x_n\}$ converges strongly to $\overline{x} = proj_{\Pi}^{f} x$.

As a direct consequence of Theorem 1 and Example 5, we obtain the convergence result in regard to Bregman totally quasi-asymptotically nonexpansive multi-valued mappings in uniformly smooth and uniformly convex Banach space. There we immediately obtain the following corollary.

Corollary 5 Let E be a uniformly smooth and uniformly convex Banach space and $f(x) = \frac{1}{p}\| x\|^p$ $(1 < p < \infty)$. Let C be a nonempty closed convex subset of $int(domf)$, $\{T_i : C \to N(C)\}_{i=1}^{\infty}$ be a countable family of closed and uniformly Bregman totally quasi-asymptotically nonexpansive multi-valued mappings with nonnegative real sequences $\{v_n\}, \{\mu_n\}$ and a strictly increasing continuous function $\zeta : R^+ \to R^+$ such that $v_n \to 0, \mu_n \to 0$ (as $n \to \infty$) and $\zeta(0) = 0$. Let $G : C \times C \to R$ be a bifunction satisfying conditions (C1)-(C4) and $\varphi : C \to R$ be a proper lower semi-continuous and convex function. Assume that T_i is uniformly L_i - Lipschitz continuous for each $i \geq 1$

and $\Omega := \bigcap_{i=1}^{\infty} F(T_i) \cap MEP(G,\varphi) \neq \varnothing$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 = x \in C, \text{chosen arbitrarily}, \\ w_n^i : G(w_n^i, y) + \varphi(y) - \varphi(w_n^i) + \langle J_p(w_n^i) - J_p(m_n^i), y - w_n^i \rangle \geq 0, \forall y \in C, m_n^i \in T_i^n x_n, i \geq 1, \\ C_n = \{z \in C : \sup_{i \geq 1} D_f(z, w_n^i) \leq D_f(z, x_n) + \xi_n\}, \\ D_n = \bigcap_{i=1}^{n} C_i, \\ x_{n+1} = proj_{D_n}^f x, \end{cases} \tag{52}$$

where $\xi_n = v_n \sup_{v \in \Omega} \zeta(D_f(v, x_n)) + \mu_n$ and $proj_{D_n}^f$ is the Bregman projection of E onto D_n . If $\Omega := \bigcap_{i=1}^{\infty} F(T_i) \cap MEP(G,\varphi)$ is bounded, then the sequence $\{x_n\}$ converges strongly to $\overline{x} = proj_{\Omega}^f x$.

Conclusion

In this paper, we could extend the strong convergence theorems for equilibrium problems and Bregman totally quasi-asymptotically nonexpansive single-valued mappings appeared in (Zhu & Huang, 2016) to the strong convergence theorems for mixed equilibrium problems and Bregman totally quasi-asymptotically nonexpansive multi-valued mappings in the setting of real reflexive Banach spaces.

Acknowledgement

The first author is thankful to the Science Achievement Scholarship of Thailand. We would like express our deep thank to Department of Mathematics, Faculty of Science, Naresuan University.

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