ส่วนขยายของกระบวนการทำซ้ำแบบ MSP

On Extension of MSP Iterative Scheme

นิฟาตมะห์ มะกาเจ และ อนิรุทธ ผลอ่อน

Nifatamah Makaje and Aniruth Phon-On

ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์ คณะวิทยาศาสตร์และเทคโนโลยี มหาวิทยาลัยสงขลานครินทร์[1]

Department of Mathematics and Computer Science, Faculty of Science and Technology, Prince of Songkla University

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บทคัดย่อ

บทความนี้เราปรับปรุงวิธีการทำซ้ำสำหรับการหาผลเฉลยของสมการไม่เชิงเส้นที่พัฒนามาจากแนวคิดการทำซ้ำแบบจุดตรึงที่มีชื่อเรียกว่าการทำซ้ำแบบ MSP แรงจูงใจสำหรับการปรับปรุงคือ เพื่อให้การคำนวณในกระบวนการทำซ้ำทำได้ง่ายขึ้น โดยการลดจำนวนครั้งในการคำนวณค่าของฟังก์ชันและหลีกเลี่ยงการหาอนุพันธ์ของฟังก์ชัน เรานำเสนอวิธีการทำซ้ำสองวิธี และได้นำเสนอผลการคำนวณเชิงตัวเลขด้วยตัวอย่างที่คัดมาจากวรรณกรรมที่ศึกษาในเรื่องการหาผลเฉลยของสมการไม่เชิงเส้น ผลลัพธ์ที่ได้พบว่าวิธีที่เรานำเสนอนั้นมีพฤติกรรมการลู่เข้าสู่คำตอบได้ดีเมื่อเปรียบเทียบกับการทำซ้ำแบบ SP และแบบ MSP ในประเด็นของการนับจำนวนรอบการทำซ้ำ

คำสำคัญ : สมการไม่เชิงเส้น, วิธีการทำซ้ำ, การทำซ้ำแบบจุดตรึง, การทำซ้ำแบบ SP, การทำซ้ำแบบ MSP

Abstract

In this paper, we modify the iterative method for solving nonlinear equations which is based on the idea of the fixed point iteration, namely MSP-iteration. The motivation is to simplify the computation via reducing the number of function evaluations and avoiding the derivative of the function. We propose two methods and illustrate the numerical results with several examples from the references for solving nonlinear equations. The results indicate that our proposed methods provide the good performance in the case iteration counting compared with SP-iteration and MSP-iteration.

Keywords : Nonlinear equations, iterative method, fixed point iteration, SP-iteration, MSP-iteration

***Corresponding author. E-mail** : nifatamah.m@psu.ac.th

Introduction

Solving nonlinear equations has been studied as one of the most importance which arises in many problems of science and engineering. The general form of a nonlinear equation is as follow equation:

$$f(x) = 0 \tag{1}$$

where $f: D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ for an interval D is a real function. One of the best known and probably the most used method for solving the preceding equation is the Newton's method. The classical Newton method is given as follows

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, 3, \dots \text{ and } f'(x_n) \neq 0. \tag{2}$$

If the initial value x_0 is close to a root of equation, the Newton's method has quadratic convergence. In another hand if a root of $f(x) = 0$ exists, it is widely used the fixed point iteration to find such root. The procedure is we first rewrite the equation $f(x) = 0$ as $x = g(x)$ and then the fixed point can be computed by iterative formula $x_{n+1} = g(x_n)$ with the suitable initial x_0. If the sequence $\{x_n\}$ converges, we found that its limit is a fixed point of function g. It then will be a root of $f(x) = 0$. Moreover if we set $g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}, f'(x_n) \neq 0$ then the Newton's method can be seen as the fixed point iteration. In the past decade, there exist numerous modifications of the Newton's method and the fixed point method which aim to improve the convergence rate or simplify the computation (Borwein & Borwein, 1991; Noor, 2000; Weerakoon& Fermando, 2000; Frontini & Sormani, 2003; Babu & Prasad, 2006; Rafiq, 2006; Mahesshwari, 2009; Phuengrattana & Suantai, 2011; Wang, 2011; Ibrahim & Murat, 2013; Kang *et al.,* 2013; Makaje & Phon-On, 2016)

In particular, in 2011 Phuengrattana and Suantai (Phuengrattana & Suantai, 2011) established three step iterative method to solve the equation $x = g(x)$. The SP-iteration was given by

$$\begin{aligned} z_n &= (1 - \gamma_n)x_n + \gamma_n g(x_n), \\ y_n &= (1 - \beta_n)z_n + \beta_n g(z_n), \\ x_{n+1} &= (1 - \alpha_n)y_n + \alpha_n g(y_n) \end{aligned} \tag{3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive real numbers in $[0, 1]$. They claimed that sequence $\{x_{n+1}\}$ converges to a fixed point of a function g. Also, under some suitable sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ the SP-iteration can be reduced to some iterative schemes and converge faster than others for the class of continuous and non-decreasing function.

In 2013, Kang et al. (Kang *et al.,* 2013) has modified the fixed point method for solving the nonlinear equation $f(x) = 0$, this can be written as $x = g(x)$. The sequence of approximated solution can be formulated as

$$x_{n+1} = \frac{-x_n g'(x_n) + g(x_n)}{1 - g'(x_n)}, \ g'(x_n) \neq 1. \tag{4}$$

In this paper they also show that sequence $\{x_{n+1}\}$ converges to a fixed point of a function g. Moreover, this iteration has a second order of convergence as the same as the order of the Newton's method. Moreover, if we set as $g(x) = f(x) + x$, then the Kang et al.-iteration become the Newton's method.

Motivated by the SP-iteration and the Kang et al.-iteration, in 2016, a new iteration method was proposed for the class of locally differentiable functions, is called MSP iteration (Makaje & Phon-On, 2016) and defined as

$$\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n g(x_n), \\ y_n &= (1-\beta_n)z_n + \beta_n g(z_n), \\ x_{n+1} &= \frac{-g'(y_n)}{1-g'(y_n)} y_n + \frac{1}{1-g'(y_n)} g(y_n) \end{aligned} \tag{5}$$

where $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive real numbers in $[0, 1]$.

The MSP method is at the least second order iterative method as the same order as the Newton's method. Although, the number of functional evaluations at each step of the MSP-iteration is greater than SP-iteration and the Newton's method but the MSP-iteration converges to a fixed point with the smaller number of iterations. Moreover, the MSP-iteration can avoid the division by near zero in the case of the initial guess close to the critical point.

According to the Kung-Traub's conjecture (Kung & Traub, 1974), an optimal iterative method based upon number of function evaluations. So in this paper we develop the iterative process in the MSP-iteration to avoid the derivative of the function and to simplify the computation but still converging with the small number of iterations.

Methods

Now we recall the MSP-iteration which was defined as

$$\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n g(x_n), \\ y_n &= (1-\beta_n)z_n + \beta_n g(z_n), \\ x_{n+1} &= \frac{-g'(y_n)}{1-g'(y_n)} y_n + \frac{1}{1-g'(y_n)} g(y_n) \end{aligned} \tag{6}$$

where $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive real numbers in $[0, 1]$.

A potential problem in implementing the MSP-iteration method is the evaluation of derivative. In practical situation, there are certain functions whose derivatives may be extremely difficult to evaluate. To avoid such difficulty, we approximate the derivative in MSP-iteration by a finite divided difference as $g'(y_n) = \frac{g(y_n) - g(z_n)}{y_n - z_n}$ where $z_n \neq y_n$. This approximation can be substituted in iteration (6) to yield the following iterative process:

$$z_n = (1-\gamma_n)x_n + \gamma_n g(x_n),$$
$$y_n = (1-\beta_n)z_n + \beta_n g(z_n), \quad (7)$$
$$x_{n+1} = \frac{g(z_n)y_n - g(y_n)z_n}{y_n - z_n - g(y_n) + g(z_n)}$$

where $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive real numbers in $[0,1]$. It will be called MSP-A.

Moreover, to respond to Kung – Traub's conjecture we reduce the number of function evaluations by reducing one step in MSP-iteration and approximate the derivative in iteration (6) to yield the following iterative process:

$$z_n = (1-\gamma_n)x_n + \gamma_n g(x_n),$$
$$x_{n+1} = \frac{g(x_n)z_n - g(z_n)x_n}{z_n - x_n - g(z_n) + g(x_n)} \quad (8)$$

where $\{\gamma_n\}$ is sequences of positive real numbers in $[0,1]$. It will be called MSP-B.

Results

In this section we illustrate the efficiency of our proposed methods (7) and (8) by comparing the number of iterations of the proposed methods with the SP-iteration and MSP-iteration. We first rewrite the nonlinear equation $f(x) = 0$ to be $x = g(x)$ and then identify the fixed point of function $g(x)$. We implement all schemes via algorithms in pseudocode using Scilab software and test them with several examples.

In (Chugh & Kumar, 2011), the authors compared the convergence rate of the SP-iteration with Picard, Mann, Ishikawa, Noor, Thianwan iterations, and claimed that the SP-iterative scheme is better than the other iterative schemes for the increasing function $2x^3 - 7x^2 + 8x - 2$ and the decreasing function $(1-x)^8$. In addition, Makaje and Phon-on (Makaje & Phon-On, 2016) considered the above functions and showed that MSP-iteration is better than SP-iteration. Hence, in the first two examples we are going to compare the numerical results by considering those two functions. To measuring the errors of the approximated solution we use the percentage relative error defined by $\varepsilon_s = \left|\frac{x_{n+1} - x_n}{x_{n+1}}\right| \times 100\%$.

Example 1 Consider $f(x) = 2x^3 - 7x^2 + 7x - 2 = 0$. The all exact solutions of $f(x) = 0$ are $x \in \{0.5, 1, 2\}$. We rewrite the equation by adding x in both sides and then obtain a function $g(x) = 2x^3 - 7x^2 + 8x - 2$. We select $x_0 = 0.8$ as an initial approximation and take $\alpha_n = \beta_n = \gamma_n = \frac{1}{(1+n)^{1/2}}$ for all iterative schemes.

Table 1 Numerical results for Example 1

iteration	SP-iteration (3)		MSP-iteration (6)		MSP-A method (7)		MSP-B method (8)	
n	$g(x_n)$	x_{n+1}	$g(x_n)$	x_{n+1}	$g(x_n)$	x_{n+1}	$g(x_n)$	x_{n+1}
0	0.944	0.9999878	0.944	1.0000124	0.944	1.000234	0.944	1.0266546
1	1.	0.9999997	1.	1.	0.9999999	1.	0.9993274	1.0004594
2	1.	1.0000000	1.	1.	1.	1.	0.9999998	1.0000002
3	1.	1.					1.	1.
4	1.	1.						

As the results are shown in Table 1, all schemes obtain the sequence of approximated solution converge to 1. If we iterate until the $\varepsilon_s < 0.5 \times 10^{-6}\%$, the SP- , MSP-iterations and proposed methods converge within 5, 3, 3, and 4 iterations, respectively.

As we mentioned, the advantage of MSP-iteration can avoid the division by near zero in the case of the initial guess close to the critical point. This would be motivated us to investigate the result of our proposed methods. Then we set $x_0 = 1.6076252$ and iterate all schemes until $\varepsilon_s < 0.5 \times 10^{-6}\%$.

For the initial guess $x_0 = 1.6076252$, the denominator in the formula of the Newton's method is closed to zero and so it will be effected on its convergence as the result shown in Table 2. However it is not oscillated on the convergence of the MSP-iteration and the proposed methods. As the results are presented in Table 3, the SP- , MSP-iterations and proposed methods converge within 5, 3, 3, and 4 iterations, respectively.

Table 2 Numerical result for Example 1 where the initial value closed to the critical point using Newton's method

n	x_n	$f(x_n)$
0	1.6076252	- 0.5281529
1	- 5392078.4	- 3.135D+20
2	- 3594718.6	- 9.290D+19
3	- 2396478.7	- 2.753D+19
⋮	⋮	⋮
30	- 41.017322	- 150082.85
⋮	⋮	⋮
45	0.5	- 7.265D-13

Table 3 Numerical results for Example 1 where the initial value closed to the critical point

iteration	SP-iteration (3)		MSP-iteration (6)		MSP-A method (7)		MSP-B method (8)	
n	$g(x_n)$	x_{n+1}	$g(x_n)$	x_{n+1}	$g(x_n)$	x_{n+1}	$g(x_n)$	x_{n+1}
0	1.079472	0.999972	1.079472	1.000029	1.079472	0.999662	1.079472	0.978475
1	1.	0.999999	0.999999	1.	0.999999	1.	0.999517	1.000156
2	1.	0.999999	1.	1.	1.	1.	1.	1.
3	1.	1.000000					1.	1.
4	1.	1.					1.	1.
5	1.	1.						

Example 2 Consider the equation $f(x) = x - (1-x)^8 = 0$. We rewrite it by algebraic manipulation and get transformed function $g(x) = (1-x)^8$. We select an initial approximation $x_0 = 0.8$ and take $\alpha_n = \beta_n = \gamma_n = \frac{1}{(1+n)^{1/2}}$ for all iterative schemes. So the results are shown as in Table 4.

Table 4 Numerical results for Example 2

iteration	SP-iteration (3)		MSP-iteration (6)		MSP-A method (7)		MSP-B method (8)	
n	$g(x_n)$	x_{n+1}	$g(x_n)$	x_{n+1}	$g(x_n)$	x_{n+1}	$g(x_n)$	x_{n+1}
0	0.000003	3.1D-38	0.000002	1.21D-32	0.000003	0.499990	0.000003	0.444442
1	1.	0.171088	1.	0.187347	0.003907	0.193761	0.009075	0.223861
2	0.222878	0.193474	0.190214	0.188347	0.178530	0.188369	0.131678	0.190116
3	0.179038	0.187960	0.188349	0.188348	0.188308	0.188348	0.185089	0.188351
4	0.189069	0.188356	0.188348	0.188348	0.188348	0.188348	0.188341	0.188348
5	0.188332	0.188348					0.188348	0.188348
6	0.188348	0.188348						
7	0.188348	0.188348						

From Table 4, all schemes converge to 0.1883477. If we iterate until the error $\varepsilon_s < 0.5 \times 10^{-6}\%$, the SP-iteration converges within 8 iterations, the MSP-iteration converges within 5 iterations and the proposed methods converge within 5 iterations and 6 iterations.

The purpose of the next example is to use all those iterative schemes to solve a transcendental equation which arises in various problems of science and engineering (Chapra & Canale, 2010).

Example 3 Consider the equation $f(\phi) = \tan(\theta) - 2\cot(\phi)\left[\frac{M^2 sin^2(\phi)-1}{M^2\,(\gamma+\cos(2\phi))+2}\right] = 0$ where $M = 3, \theta = \frac{\pi}{9}$, and $\gamma = 1.4$. We rewrite the equation by adding ϕ in both sides and then obtain a transformed function $g(\phi) = \phi + \tan(\theta) - 2\cot(\phi)\left[\frac{M^2 sin^2(\phi)-1}{M^2\,(\gamma+\cos(2\phi))+2}\right]$. We select $\phi_0 = \frac{\pi}{4}$ as an initial approximation and take $\alpha_n = \frac{1}{(1+n)^{1/2}}, \beta_n = \gamma_n = \frac{1}{(1+n)^2}$ for all those iterative schemes.

Table 5 Numerical results for Example 3

Method	n	ϕ_{n+1}	$f(\phi_{n+1})$	ε_s
SP-iteration (3)	7	0.6590998	- 6.620D-08	0.0000059
	14	0.6590998	- 5.965D-09	0.0000003
MSP-iteration (6)	2	0.6590998	1.665D-16	1.011D-13
MSP-A method (7)	2	0.6590998	2.220D-16	8.722D-11
MSP-B method (8)	2	0.6590998	6.106D-16	0.0000125
Newton's method	2	0.6590998	1.274D-10	0.0028837
	3	0.6590998	2.220D-16	2.014D-08

If we restrict $|f(x_{n+1})| < 0.5 \times 10^{-6}\%$ so the SP-iteration converges in 8 iterations while the MSP-iteration, the proposed methods, and the Newton's method converge within 3 iterations. For the MSP-B method (8), because the divisor $z_1 - x_1$ equals to 0, the computation has to be stopped at $n = 2$. However, the value of $|f(x_{n+1})|$ is closely to zero. Moreover, if we consider $\varepsilon_s < 0.5 \times 10^{-6}\%$ then SP-iteration converges in 15 iterations, the Newton method converges within 4 iterations, and the MSP-iteration and the MSP-A method (7) converge within 3 iterations.

Discussion

From Tables 1-5, we see that the number of iterations of the MSP-A method (7) is equal to the number of iterations of the MSP-iteration, while the number of iteration of the MSP-B method (8) is greater than that of the MSP-iteration by one. In addition, in the case of the initial guess close to the critical point the number of iterations of MSP-A and MSP-B method are less than the Newton's method as shown in Example 1. Furthermore, if we consider methods which is not required to compute the derivative of function, we found that the MSP-A method (7) and the MSP-B method (8) converge faster than the SP-iteration.

Conclusions

In this paper, we presented the extension of the MSP iteration in order to simplify the computation in the procedure. We developed the iterative process in MSP-iteration which includes two particular cases, avoiding the derivative of the function and reducing the number of function evaluations. From numerical examples, we observed that the proposed methods show at least the same performance as that of other known methods but the advantages of the proposed method is that in each step it requires only two or three evaluations of function and no derivatives of function required. The comparison of the computationally conditions of each method has shown in Table 6.

Table 6 Comparison the computationally conditions

	Number of control sequence	Function evaluation in each iteration	Derivative	Divisor by zero
SP-iteration	3	3	no	no
MSP-iteration	2	4	yes	yes
MSP-A method	2	3	no	yes
MSP-B method	1	2	no	yes
Newton's method	0	2	yes	yes

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