จำนวนของจัตุรัสกลขนาด 3×3 บางชนิดและขั้นตอนวิธีการสร้าง

The Number of a Type of 3×3 Magic Squares and Its Construction Algorithm

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ในงานวิจัยนี้เราแสดงสูตรการนับจำนวนของจัตุรัสกลขนาด 3×3 ชนิดหนึ่ง ซึ่งเติมด้วยจำนวนเต็มที่ไม่เป็นลบ นอกจากนี้ยังได้แสดงขั้นตอนวิธีอย่างง่ายในการสร้างจัตุรัสกลดังกล่าวทั้งหมดอีกด้วย

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Abstract

In this paper, we give a formula to count the exact number of a special type of 3×3 magic squares of nonnegative integers. We also present a simple algorithm to construct such magic squares.

Keywords : Magic square, Magic constant

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Introduction

There are various types of magic squares and one of the well-known is normal magic square which is a square matrix of order n^2 whose entries are distinct numbers from 1 to n^2 such that the sum of the numbers in every row, in every column, and in each diagonal is the same number, called the magic constant, which depends only on n and has the value $\frac{n(n^2+1)}{2}$. Numerous papers have been written about finding the number and constructing many types of magic squares. For example, normal magic squares exist for all orders n, except $n = 2$, there is only one normal magic square of order 3 (does not include rotations and reflections), total number of possible 4×4 normal magic squares is 880, etc. (Sallows, 2013; Sierpinski, 1988).

In 1997, Bona presented a new proof of a formula to find the number of a class of 3×3 magic squares in which all entries are nonnegative integers with repetition allowed, and all row sums and column sums have the same number r. He showed that the number is equal to the sum of three binomial coefficients as follows $\binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4}$ (Bona, 1997).

In 2008 Xin studied a class of 3×3 magic squares which all entries are distinct nonnegative integers such that every row sum, column sum, and two diagonal sum are equal. He showed that this type of magic squares can be generated by three basis elements (Xin, 2008).

In this paper, our objective is to study a class of 3×3 magic squares which lies between those two classes of magic squares studied by Bona and Xin. For any given nonnegative integers k and r, we denote by $M_3(k,r)$ the set of all 3×3 magic squares defined by:

(i) all entries are greater than or equal to k with repetition allowed,

(ii) the sums along rows, columns, and main diagonal (all the entries from the upper left corner to the lower right corner) are all equal to r.

By the above definition, the sum in the anti-diagonal (all the entries from the lower left corner to the upper right corner) does not necessarily equal r and it is clear that $M_3(k,r)$ is a subclass of magic squares studied in 1997 which every line sum is equal to r. In particular, when $k = 0$, then $M_3(0,r)$ contains all magic squares studied in 2008 which every line sum is equal to r.

In Section 3, we give a formula to count the exact number of $M_3(k,r)$ for any given nonnegative integers k and r. Then, in Section 4 we present an algorithm for constructing such magic squares.

Methods

By the definition of $M_3(k,r)$, it is clear that the three suitable elements $a_{11} = a, a_{22} = b$ and $a_{31} = c$ in Figure 1 below determine all the rest of the square.

$$\begin{bmatrix} a & \cdots & \cdots \\ \cdots & b & \cdots \\ c & \cdots & \cdots \end{bmatrix} \to \begin{bmatrix} a & r-a-2b+c & 2b-c \\ r-a-c & b & a-b+c \\ c & a+b-c & r-a-b \end{bmatrix}$$

Figure 1 generating elements of each magic square in $M_3(k,r)$.

We also see that, if $A = [a_{ij}]_{3\times3}$ and $B = [b_{ij}]_{3\times3}$ are two magic squares in $M_3(k,r)$ for which $a_{11} = b_{11}, a_{22} = b_{22}$ and $a_{31} = b_{31}$ then $A = B$. This means that each triple $(a,b,c) = (a_{11}, a_{22}, a_{31})$ generates only one magic square in $M_3(k,r)$. Therefore, to determine the cardinality of $M_3(k,r)$, we only need to compute the number of ways to choose the triples $(a,b,c) = (a_{11}, a_{22}, a_{31})$. We begin with the following lemma which describes the structure of the magic squares in $M_3(k,r)$ and it will be useful for proving our main theorem.

Lemma 1. Let $A = [a_{ij}]_{3\times3}$ be a magic square in $M_3(k,r)$. Then either

(a) $a_{13} = a_{22} = a_{31}$, or

(b) a_{13}, a_{22} and a_{31} are all different, in this case, either

(b1) $a_{31} < a_{22} < a_{13}$ and $a_{13} = a_{22} + l,\ a_{31} = a_{22} - l$ for some positive integer l, or

(b2) $a_{13} < a_{22} < a_{31}$ and $a_{13} = a_{22} - l,\ a_{31} = a_{22} + l$ for some positive integer l.

Proof. Assume that (b) does not hold. Thus, there are at least two numbers from the set $\{a_{13}, a_{22}, a_{31}\}$ which are equal. We suppose that $a_{13} = a_{22} = p$. Then, we can compute $a_{21} = r - a_{11} - a_{31}$, $a_{12} = r - a_{11} - p = a_{33}$ and $a_{32} = a_{11} = a_{23}$. Thus,

$$A = \begin{bmatrix} a_{11} & r-a_{11}-p & p \\ r-a_{11}-a_{31} & p & a_{11} \\ a_{31} & a_{11} & r-a_{11}-p \end{bmatrix}. \qquad (1)$$

Now, we consider the sum in the third row, we have $a_{31} + a_{11} + (r - a_{11} - p) = r$, which implies $a_{31} = p$. Therefore, $a_{13} = a_{22} = a_{31}$. In a similar way, we can prove that if $a_{13} = a_{31}$ or $a_{22} = a_{31}$, then $a_{13} = a_{22} = a_{31}$. Hence, (a) holds.

In particular, if a_{13}, a_{22} and a_{31} are all different, then we consider the following cases:

Case 1: $a_{22} < a_{13}$, say $a_{13} = a_{22} + l$ for some positive integer l. We first calculate $a_{33} = r - a_{11} - a_{22}$ followed by calculating $a_{23}, a_{21}, a_{31}, a_{32}$ and a_{12} respectively. Then we get the magic square as follows:

$$A = \begin{bmatrix} a_{11} & r - a_{11} - a_{22} - l & a_{22} + l \\ r - a_{11} - a_{22} + l & a_{22} & a_{11} - l \\ a_{22} - l & a_{11} + l & r - a_{11} - a_{22} \end{bmatrix}, \quad (2)$$

that is, (b1) holds.

Case 2: $a_{13} < a_{22}$, say $a_{13} = a_{22} - l$ for some positive integer l. We use a similar argument as in Case 1, our magic square can be written as follows:

$$A = \begin{bmatrix} a_{11} & r - a_{11} - a_{22} + l & a_{22} - l \\ r - a_{11} - a_{22} - l & a_{22} & a_{11} + l \\ a_{22} + l & a_{11} - l & r - a_{11} - a_{22} \end{bmatrix}. \quad (3)$$

Then, our magic square satisfies (b2), which completes the proof. □

In view of Lemma 1, when the anti-diagonal entries are equal, the magic square in (1) can be clearly seen as

$$A = \begin{bmatrix} a_{11} & r - a_{11} - p & p \\ r - a_{11} - p & p & a_{11} \\ p & a_{11} & r - a_{11} - p \end{bmatrix}, \quad (4)$$

where its reflection over the main diagonal is itself. On the other hand, when the entries on the anti-diagonal are all different, we get the magic squares represented in (2) and (3), where they are the reflection of each other over the main diagonal.

Results

Before proving the main result, we observe that if $A = [a_{ij}]_{3\times 3}$ is a magic square in $M_3(k,r)$, then $a_{ij} \geq k$ for all i,j by its definition. Thus, each entries can be written as $k + x$ for some non-negative integer x. Consequently, if $k + x, k + y$ and $k + z$ are entries in the same row (column, or the main diagonal) of the magic square, then we have $(k + x) + (k + y) + (k + z) = r$ which implies $x + y + z = r - 3k$ and, thus, $0 \leq x \leq r - 3k$. Therefore, the magic squares in $M_3(k,r)$ can be constructed only when $0 \leq r - 3k$.

In what follows, for any real number x, let $\lfloor x \rfloor$ denote the largest integer less than or equal to x. We also note that the summation of the first n natural numbers, i.e. $1 + 2 + 3 + ... + n$, is equal to $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Now, we are in the position to prove our main theorem.

Theorem 1. Let k and r be nonnegative integers. Then the number of magic squares in $M_3(k,r)$ is

$$(s+1)^2-\frac{s(s+1)}{2}+\sum_{l=1}^{\lfloor \frac{s}{3} \rfloor}((s-3l+1)^2+(s-3l+1))$$

where $s=r-3k$.

Proof. Recall that, once we know the triple (a_{11},a_{22},a_{31}), we can determine each magic square in $M_3(k,r)$, so our aim is to count the number of ways to choose these triples. From Lemma 1, we first count all magic squares in (4) which all entries in their anti-diagonal are all equal. We write $p=k+x$ and $a_{11}=k+y$ for some $0\le x,y\le r-3k$, then (4) can be written as

$$A=\begin{bmatrix} k+y & r-x-y-2k & k+x \\ r-x-y-2k & k+x & k+y \\ k+x & k+y & r-x-y-2k \end{bmatrix}. \tag{5}$$

Consider a_{21} in (5), we have that $r-x-y-2k\ge k$ which implies that $0\le y\le r-3k-x$. So, $a_{11}=k+y$ can be chosen in $r-3k-x+1$ ways and this number depends on x, where $0\le x\le r-3k$. Therefore, the number of ways to choose the triples $(a_{11},a_{22},a_{31})=(k+y,k+x,k+x)$ is

$$\begin{aligned}\sum_{x=0}^{r-3k}(r-3k-x+1)&=\sum_{x=0}^{r-3k}(r-3k+1)-\sum_{x=0}^{r-3k}x \\ &=(r-3k+1)^2-\frac{(r-3k)(r-3k+1)}{2}.\end{aligned} \tag{6}$$

This shows the number of magic squares for which all entries in the anti-diagonal are the same.

Next, we count all magic squares in the form of (2), i.e., $a_{31}<a_{22}<a_{13}$. We write $a_{11}=k+t$ and $a_{22}=k+u$ for some $0\le t,u\le r-3k$. Then (2) can be written as

$$A=\begin{bmatrix} k+t & r-t-u-2k-l & k+u+l \\ r-t-u-2k+l & k+u & k+t-l \\ k+u-l & k+t+l & r-t-u-2k \end{bmatrix}. \tag{7}$$

Now, all entries of the magic square in (7) must be greater than or equal to k by the definition. But, in fact, we only need three following conditions:

$$a_{12} = r - t - u - 2k - l \geq k, \tag{8}$$

$$a_{23} = k + t - l \geq k, \tag{9}$$

$$a_{31} = k + u - l \geq k. \tag{10}$$

This is because the conditions $a_{11} \geq k$ and $a_{32} \geq k$ are implied by (9) (as $a_{32} > a_{11} > a_{23} \geq k$). Similarly, $a_{13} \geq k$ and $a_{22} \geq k$ are implied by (10), and $a_{21} \geq k$, $a_{33} \geq k$ are implied by (8). Thus, we can only focus on the conditions (8), (9) and (10). Now, conditions (8) and (10) imply

$$l \leq u \leq r - 3k - t - l, \tag{11}$$

which means that there are $(r - 3k - t - l) - l + 1 = r - 3k - t - 2l + 1$ ways to choose u and this number depends on the two variables t and l . By (9) and (11), we get that

$$l \leq t \leq r - 3k - 2l. \tag{12}$$

Next, combine (8),(9) and (10) together, we have $r - 3l \geq 3k$ which implies

$$1 \leq l \leq \left\lfloor \frac{r - 3k}{3} \right\rfloor. \tag{13}$$

From (11),(12) and (13), the number of ways to choose all triples $(a_{11}, a_{22}, a_{31}) = (k + t, k + u, k + u - l)$ which generate all magic squares of the form (7) is

$$\sum_{l=1}^{\left\lfloor \frac{r-3k}{3} \right\rfloor} \sum_{t=l}^{r-3k-2l} (r - 3k - t - 2l + 1), \tag{14}$$

where the inner summation in (14) can be expressed as

$$(r - 3k - l - 2l + 1) + (r - 3k - (l + 1) - 2l + 1) + (r - 3k - (l + 2) - 2l + 1) + \ldots$$

$$+ (r - 3k - (r - 3k - 2l) - 2l + 1)$$

$$= (r - 3k - 3l + 1) + (r - 3k - 3l) + (r - 3k - 3l - 1) + \ldots + 1$$

$$= \frac{(r - 3k - 3l + 1)((r - 3k - 3l + 1) + 1)}{2}$$

$$= \frac{(r - 3k - 3l + 1)^2 + (r - 3k - 3l + 1)}{2}.$$

Hence, from (14), the number of magic squares of type (2) is

$$\sum_{l=1}^{\left\lfloor \frac{r-3k}{3} \right\rfloor} \frac{(r - 3k - 3l + 1)^2 + (r - 3k - 3l + 1)}{2}. \tag{15}$$

Finally, as discussed after Lemma 1, all the magic squares of type (3), i.e., $a_{13} < a_{22} < a_{31}$, are the reflection of all the magic squares of type (2), so they both have the same number of magic squares as shown in (15). By combining the results from (6) and (15) and take $s = r - 3k$, we obtain that

$$\begin{aligned} |M_3(k,r)| &= (s+1)^2 - \frac{s(s+1)}{2} + 2\cdot\sum_{l=1}^{\left\lfloor \frac{s}{3} \right\rfloor} \frac{(s-3l+1)^2 + (s-3l+1)}{2} \\ &= (s+1)^2 - \frac{s(s+1)}{2} + \sum_{l=1}^{\left\lfloor \frac{s}{3} \right\rfloor} ((s-3l+1)^2 + (s-3l+1)), \end{aligned} \tag{16}$$

which completes the proof. □

Example 1. Let $k = 5, r = 17$. Then, $s = 2$ and $\left\lfloor \frac{s}{3} \right\rfloor = 0$. Hence, the summation in the formula (16) is equal to 0, which means that there is no magic square in $M_3(5,17)$ for which $a_{31} < a_{22} < a_{13}$ or $a_{13} < a_{22} < a_{31}$.

Therefore, $|M_3(5,17)| = 3^2 - \frac{2\cdot 3}{2} = 6$.

Example 2. Let $k = 2, r = 13$. Then, $s = 7$ and $\left\lfloor \frac{s}{3} \right\rfloor = 2$. Therefore,

$$|M_3(2,13)| = 8^2 - \frac{7\cdot 8}{2} + ((5^2+5) + (2^2+2)) = 36 + (30+6) = 72.$$

Discussion

In this section, we present a simple algorithm to construct all magic squares in $M_3(k,r)$. Refer to the proof of Theorem 1, forming all magic squares in $M_3(k,r)$ can be done using the following steps:

Step (i) Calculate all values x from $0 \le x \le r - 3k$, and for each such x, calculate all values y from $0 \le y \le r - 3k - x$.

Step (ii) Write all triples $(a_{11}, a_{22}, a_{31}) = (k+y, k+x, k+x)$ from x and y in Step (i), then, all magic squares whose anti-diagonal entries are equal can be formed by these triples.

Step (iii) Calculate all values l from $1 \le l \le \left\lfloor \frac{r-3k}{3} \right\rfloor$, and for each such l, calculate all values t from $l \le t \le r - 3k - 2l$.

Step (iv) Find all values u satisfying $l \le u \le r - 3k - t - l$ from l, t in Step (iii).

Step (v) Write all triples $(a_{11}, a_{22}, a_{31}) = (k+t, k+u, k+u-l)$ from l, t and u from Step (iii) and Step (iv), then we can use these triples to generate all magic squares in which $a_{31} < a_{22} < a_{13}$.

Step (vi) All magic squares in which $a_{13} < a_{22} < a_{31}$ can be created from the reflection of all those magic squares in Step (v) to the main diagonal, where their generating triples (a_{11}, a_{22}, a_{31}) are of the form $(k+t, k+u, k+u+l)$.

Example 3. From Example 1, we compute $|M_3(5,17)| = 6$ .Table 1. below shows all 6 triples $(a_{11}, a_{22}, a_{31}) = (k+y, k+x, k+x)$ which generate all magic squares in $M_3(5,17)$.

Table 1 all generating triples for magic squares in $M_3(5,17)$.

$0 \le x \le r-3k$	$0 \le y \le r-3k-x$	$(k+y, k+x, k+x)$
0	0	(5,5,5)
	1	(6,5,5)
	2	(7,5,5)
1	0	(5,6,6)
	1	(6,6,6)
2	0	(5,7,7)

Example 4. From Example 2, we compute $|M_3(2,13)| = 72$. The Table 2 below shows all 36 triples which generate all magic squares in $M_3(2,13)$ for which $a_{13} = a_{22} = a_{31}$ and Table 3 shows all 36 triples which generate all magic squares in $M_3(2,13)$ for which $a_{31} < a_{22} < a_{13}$ (see column 4) and $a_{13} < a_{22} < a_{31}$ (see column 5).

Table 2 36 generating triples for magic squares in $M_3(2,13)$ which $a_{13} = a_{22} = a_{31}$.

$0 \le x \le r-3k$	$0 \le y \le r-3k-x$	$(k+y, k+x, k+x)$
0	0	(2,2,2)
	1	(3,2,2)
	2	(4,2,2)
	3	(5,2,2)
	4	(6,2,2)
	5	(7,2,2)
	6	(8,2,2)
	7	(9,2,2)
	0	(2,3,3)

$0 \le x \le r-3k$	$0 \le y \le r-3k-x$	$(k+y, k+x, k+x)$
1	1	(3,3,3)
	2	(4,3,3)
	3	(5,3,3)
	4	(6,3,3)
	5	(7,3,3)
	6	(8,3,3)
2	0	(2,4,4)
	1	(3,4,4)
	2	(4,4,4)
	3	(5,4,4)
	4	(6,4,4)
	5	(7,4,4)
3	0	(2,5,5)
	1	(3,5,5)
	2	(4,5,5)
	3	(5,5,5)
	4	(6,5,5)
4	0	(2,6,6)
	1	(3,6,6)
	2	(4,6,6)
	3	(5,6,6)
5	0	(2,7,7)
	1	(3,7,7)
	2	(4,7,7)
6	0	(2,8,8)
	1	(3,8,8)
7	0	(2,9,9)

Next, we show all generating triples for magic squares in $M_3(2,13)$ which a_{13}, a_{22}, a_{31} are all distinct.

Table 3 36 generating triples for magic squares in $M_3(2,13)$ which $a_{31} < a_{22} < a_{13}$ (see column 4) and $a_{13} < a_{22} < a_{31}$ (see column 5).

$1 \le l \le \left\lfloor \frac{r-3k}{3} \right\rfloor$	$l \le t \le r-3k-2l$	$l \le u \le r-3k-t-l$	$(k+t, k+u, k+u-l)$	$(k+t, k+u, k+u+l)$
1	1	1	(3,3,2)	(3,3,4)
		2	(3,4,3)	(3,4,5)
		3	(3,5,4)	(3,5,6)
		4	(3,6,5)	(3,6,7)
		5	(3,7,6)	(3,7,8)
	2	1	(4,3,2)	(4,3,4)
		2	(4,4,3)	(4,4,5)
		3	(4,5,4)	(4,5,6)
		4	(4,6,5)	(4,6,7)
	3	1	(5,3,2)	(5,3,4)
		2	(5,4,3)	(5,4,5)
		3	(5,5,4)	(5,5,6)
	4	1	(6,3,2)	(6,3,4)
		2	(6,4,3)	(6,4,5)
	5	1	(7,3,2)	(7,3,4)
2	2	2	(4,4,2)	(4,4,6)
		3	(4,5,3)	(4,5,7)
	3	2	(5,4,2)	(5,4,6)

For example, a magic square in $M_3(2,13)$ generated by (5,4,2) can be formed as follows:

$$\begin{bmatrix} 5 & \cdots & \cdots \\ \cdots & 4 & \cdots \\ 2 & \cdots & \cdots \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 2 & 6 \\ 6 & 4 & 3 \\ 2 & 7 & 4 \end{bmatrix}$$

Figure 2 constructing a magic square in from (5,4,2).

Conclusions

In this paper, a class of 3×3 magic squares denoted by $M_3(k,r)$ has been examined, a formula to count the number of all magic squares in $M_3(k,r)$ has been presented as well as an algorithm for constructing such magic squares. We have shown that such number equal to the number of ways to choose the possible number to fill in three entries a_{11}, a_{22}, a_{31} in the magic square.

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