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เซตปิดปกติวางนัยทั่วไปเทียบกับอุดมคติ

Regular Generalized Closed Sets with Respect To an Ideal

พรหมลิขิต จิตจักร[1*] และ พันธ์ทิพา คนฉลาด[2]

Promlikhit Jitjar[1*] and Phuntipa Konchalard[2]

[1]*กลุ่มสาระการเรียนรู้คณิตศาสตร์ โรงเรียนนครพนมวิทยาคม*

[2]*สาขาวิชาคณิตศาสตร์ คณะครุศาสตร์ มหาวิทยาลัยราชภัฏร้อยเอ็ด*

[1]*Mathematics, Nakhon Phanom Witthayakhom School*

[2]*Department of Mathematics, Faculty of Education, Roi Et Ratchaphat University*

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บทคัดย่อ

ในบทความนี้ เราแนะนำและศึกษาแนวคิด เซต μ -ปิดปกติวางนัยทั่วไปเทียบกับอุดมคติและเซต μ -เปิดปกติวางนัยทั่วไปเทียบกับอุดมคติและศึกษาสมบัติบางประการ

คำสำคัญ : ปริภูมิทอพอโลยีวางนัยทั่วไปอุดมคติ ; เซตปิด $I\mu$ -rg ; เซตเปิด $I\mu$ -rg

Abstract

In this paper, we introduce and study the notions of μ -regular generalized closed sets with respect to an ideal and μ -regular generalized closed sets with respect to an ideal and study their properties.

Keywords : ideal generalized topological space ; $I\mu$ -rg -closed set ; $I\mu$ -rg -open set.

* Corresponding author. Email : promlikhit18@gmail.com

Introduction

The notions of ideal in topological spaces have been studied by Kuratowski (1933). In 2002, Császár studied generalized topological spaces. Such generalized topological spaces in the sense of Császár (2005) are a generalization of topological spaces (briefly GTS), that consist of X and structure μ on X (briefly GT) such that μ is closed under arbitrary unions. Then (X, μ) is called a generalized topological space. He also introduced closure (c_μ) and interior (i_μ) in generalized topological spaces. Later the concepts of g-closed sets in topological spaces were extended to generalized topological spaces (X, μ) by Roy (2011). Jitjagr *et al.,* (2015) introduced the concepts of μ-regular generalized closed sets (briefly μrg-closed sets) and μ-regular generalized open sets (briefly μrg-open sets) in generalized topological spaces. In 2016, Modak introduced the notions of the ideal generalized topological spaces and to investigate the relationships between generalized topological spaces and ideal generalized topological spaces. He obtained some properties of generalized topological space and ideal generalized topological space.

In this paper, we introduce the notions of a μ-regular generalized closed sets with respect to an ideal and μ-regular generalized open sets with respect to an ideal. Moreover, some properties of such sets are obtained.

Preliminaries

In this paper, we begin by recalling the notion of generalized topology (briefly, GT) Császár (2002), μ on a non-empty set X is a collection of subset of X such that $\varnothing \in \mu$ and an arbitrary union of elements of μ belongs to μ. A set X with a GT μ is called a generalized topological space (briefly, GTS). A subset A of X is called μ-open if $A \in \mu$. The complement of a μ-open set is called μ-closed set. For a GTS (X, μ) and $A \subset X$ then the μ-closure of A Császár (2005), $c_\mu(A)$ is the intersection of all μ-closed sets containing A and the μ-interior of A, $i_\mu(A)$, is the of all μ-open sets contained in A. A GTS (X, μ) is called quasi-topological space (briefly, QTS) Császár (2008), and we call μ a quasi-topology (briefly, QT), if any finite intersection of elements of μ belongs to μ. If (X, μ) is QTS and $A, B \subset X$, then $c_\mu(A \cup B) = c_\mu(A) \cup c_\mu(B)$ and $i_\mu(A \cup B) = i_\mu(A) \cup i_\mu(B)$ Jamunarani and Jeyanthi (2012).

The ideals I on non-empty set X is a non-empty collection of subsets of X which satisfy the following properties : (i) $A \in I$ and $B \subseteq A$ imply $B \in I$ and (ii) $A \in I$ and $B \in I$ imply $A \cup B \in I$. An ideal generalized topological space (X, μ) with an ideal I on X is denoted by (X, μ, I) Modak (2016).

The following is useful in the sequel.

Theorem 1 [Császár (2002)]. Let (X, μ) be a generalized topological space and $A \subset X$. Then

(i) $c_\mu(A) = X - i_\mu(X - A)$,

(ii) $i_\mu(A) = X - c_\mu(X - A)$.

Lemma 2 [Navaneethakrishnan and Sivaraj (2010)]. Let A be a subset of a generalized topological space (X, μ). Then A is

(i) μ-regular open if and only if $A = i_\mu(c_\mu(A))$,

(ii) μ-regular open if and only if $A = i_\mu(B)$ for some μ-closed set B,

(iii) μ-regular closed if and only if $A = c_\mu(B)$ for some μ-open set B.

Theorem 3 [Navaneethakrishnan and Sivaraj (2010)]. Let (X, μ) be a generalized topological space. Then the following are equivalent:

(i) Every μ-regular open set of X is μ-closed.

(ii) Every subset of X is μ-rg-closed.

Definition 4 [Sarsak (2012)]. Let A be a subset of a generalized topological space (X, μ). Then A is called

(i) μ-regular closed if and only if $A = c_\mu(i_\mu(A))$,

(ii) μ-regular open if and only if $X - A$ is μ-regular closed.

Definition 5 [Jitjagr *et al.*, (2015)]. Let (X, μ) be a generalized topological space and $A \subset X$. Then A is called a μ-regular generalized closed (briefly, μrg-closed) set if and only if $c_\mu(A) \subset U$ whenever $A \subset U$ and U is a μr-open set in (X, μ).

Methods

The research procedure consists of the following step:

1 Criticism and the possible extension of the literature review.

2 Researching to investigate the results.

3 Applying the results from 1 and 2 to the results.

Results

In this section, we introduce μ -regular generalized closed set with respect to an ideal and investigate some of their properties.

μ -regular generalized closed sets with respect to an ideal

Definition 6 Let (X, μ, I) be an ideal generalized topological space.
A subset A of X is said to be a μ -regular generalized closed set with respect to I (briefly $I\mu$ -rg -closed set) if $c_\mu(A) - U \in I$ whenever $A \subset U$ and U is μ -regular open.

Remark 7 For an ideal generalized topological space, every μrg -closed set is $I\mu$ -rg -closed, but the converse need not be true, as this may be seen from the following example.

Example 8 Let $X = \{a, b, c\}$ with a generalized topology $\mu = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$ and $I = \{\varnothing, \{b\}, \{c\}, \{b, c\}\}$. Then (X, μ, I) is an ideal generalized topological space. Moreover, $\{\varnothing, \{a\}, \{b\}, \{a, b\}\}$ is the set of all μ -regular open sets in X. Take $A = \{a, b\}$ .Then A is $I\mu$ -rg -closed because $\{a, b\}$ is the only μ -regular open containing A and $c_\mu(A) - \{a, b\} = X - \{a, b\} = \{c\} \in I$. But A is not μ -rg -closed because $c_\mu(A) = X \not\subset \{a, b\}$ and $\{a, b\}$ is μ -regular open.

Theorem 9 Let (X, μ, I) be an ideal generalized topological space and let $A, B \subset X$. If A is $I\mu$ -rg -closed and $A \subset B \subset c_\mu(A)$, then B is $I\mu$ -rg -closed.

Proof. Assume that A is $I\mu$ -rg -closed and $A \subset B \subset c_\mu(A)$. Let U be a μ -regular open such that $B \subset U$. Then $A \subset U$. Since A is a $I\mu$ -rg -closed set, we have $c_\mu(A) - U \in I$. Now $B \subset c_\mu(A)$. This implies that $c_\mu(B) - U \subset c_\mu(A) - U \in I$. Hence B is $I\mu$ -rg -closed.

Definition 10 A generalized topological space (X, μ) is called μ -locally indiscrete if every μ -closed set is μ -regular closed.

Example 11 Let $X = \{a, b, c\}$ with a generalized topology $\mu = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$. Then $X, \{b, c\}, \{a, c\}, \{c\}$ are μ -closed sets. Moreover, $X, \{b, c\}, \{a, c\}, \{c\}$ are μ -regular closed. Hence (X, μ) is μ -locally indiscrete.

Theorem 12 Let (X, μ, I) be an ideal generalized topological space where (X, μ) is

μ -locally indiscrete. Then a subset A of X is $I\mu$ -rg -closed if and only if $F \subset c_\mu(A) - A$ and F is μ -regular closed in X implies $F \in I$.

Proof. Assume that A is $I\mu$ -rg -closed. Let F is μ -regular closed set such that $F \subset c_\mu(A) - A$. Then $A \subset X - F$ where $X - F$ is μ -regular open. By assumption, $c_\mu(A) - (X - F) \in I$. But $F \subset c_\mu(A)$, $F = c_\mu(A) \cap F = c_\mu(A) - (X - F)$, and hence $F \in I$.

Conversely, assume that $F \subset c_\mu(A) - A$ and F is μ -regular closed in X implies that $F \in I$. Suppose that $A \subset U$ and U is μ -regular open. Since $c_\mu(A)$ and $X - U$ are μ -closed, $c_\mu(A) - U = c_\mu(A) \cap (X - U)$ is μ -closed. Since (X, μ) is μ -locally indiscrete, $c_\mu(A) \cap (X - U)$ is a μ -regular closed set in X , that is contained in $c_\mu(A) - A$. By assumption, $c_\mu(A) - U \in I$. This implies that A is $I\mu$ -rg -closed.

In general, the union of two $I\mu$ -rg -closed set need not be a $I\mu$ -rg -closed set as seen from the next example.

Example 13 Consider the generalized topological space (X, μ) where $X = \{a, b, c, d\}$ with a generalized topology $\mu = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, X\}$ and $I = \{\varnothing, \{b\}, \{c\}, \{bc\}\}$. Then (X, μ, I) is an ideal generalized topological space. Moreover, $\{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c, d\}, \{b, c, d\}, X\}$ is the set of all μ -regular open sets in X . Then $\{a\}, \{b\}, \{a, c, d\}, \{b, c, d\}$ and X are $I\mu$ -rg -closed but their union $\{a\} \cup \{b\} = \{a, b\}$ is not $I\mu$ -rg -closed.

Theorem 14 Let (X, μ, I) be an ideal generalized topological space where μ is a quasi-topology and let $A, B \subset X$. If A and B are $I\mu$ -rg -closed sets, then their union $A \cup B$ is also $I\mu$ -rg -closed.

Proof. Suppose that A and B are $I\mu$ -rg -closed sets. If $A \cup B \subset U$ and U is μ -regular open, then $A \subset U$ and $B \subset U$. Since A and B are $I\mu$ -rg -closed, $c_\mu(A) - U \in I$ and $c_\mu(B) - U \in I$. Hence $(c_\mu(A) - U) \cup (c_\mu(B) - U) \in I$. Since μ is a quasi-topology, $c_\mu(A \cup B) = c_\mu(A) \cup c_\mu(B)$, and hence $c_\mu(A \cup B) - U = (c_\mu(A) - U) \cup (c_\mu(B) - U) \in I$. Therefore, $A \cup B$ is $I\mu$ -rg -closed.

Theorem 15 Let (X, μ, I) be an ideal generalized topological space where every μ -regular open set is μ -closed. Then every subset of X is $I\mu$ -rg -closed.

Proof. Let A be a subset of X and U be μ -regular open in X , such that $A \subset U$. Since every μ -regular open set is μ -closed, $c_\mu(U) = U$. Thus $c_\mu(A) \subset c_\mu(U) = U$. Hence $c_\mu(A) - U = \varnothing \in I$. Therefore, A is $I\mu$ -rg -closed.

μ -regular generalized open sets with respect to an ideal

In this section, we introduce and study μ -regular generalized open sets with respect to an Ideal and investigate some of their properties.

Definition 16 Let (X, μ, I) be an ideal generalized topological space. A subset A of X is said to be a μ -regular generalized open set with respect to I (briefly $I\mu$ -rg -open set) if $X - A$ is $I\mu$ -rg -closed.

Theorem 17 Let (X, μ, I) be an ideal generalized topological space. A subset A of X is $I\mu$ -rg -open if and only if $F - U \subset i_\mu(A)$, for some $U \in I$, whenever F is a μ -regular closed set such that $F \subset A$.

Proof. Suppose that A is $I\mu$ -rg -open. Let $F \subset A$ and F is μ -regular closed. We have $X - A \subset X - F$ and $X - F$ is μ -regular open. Since $X - A$ is $I\mu$ -rg -closed, $c_\mu(X - A) - (X - F) \in I$. Set $U = c_\mu(X - A) - (X - F)$. Then $U \in I$ and $c_\mu(X - A) \subset (X - F) \cup U$. This implies $X - ((X - F) \cup U) \subset X - c_\mu(X - A)$. Since $X - ((X - F) \cup U) = F - U$ and $X - c_\mu(X - A) = i_\mu(A)$, $F - U \subset i_\mu(A)$.

Conversely, assume that $F \subset A$ and F is μ -regular closed set imply $F - U \subset i_\mu(A)$, for some $U \in I$. Let G be a μ -regular open set, such that $X - A \subset G$. This implies $X - G$ is μ -regular closed and $X - G \subset A$. By assumption, $(X - G) - U \subset i_\mu(A)$ for some $U \in I$. Thus $X - (G \cup U) \subset X - c_\mu(X - A)$ .Then $c_\mu(X - A) \subset G \cup U$. This show that $c_\mu(X - A) - G \subset (G \cup U) - G \subset U \in I$. Therefore, we have $X - A$ is $I\mu$ -rg -closed. Hence A is $I\mu$ -rg -open.

Recall that the sets A and B are said to be μ -separated if $c_\mu(A) \cap B = \varnothing$ and $c_\mu(B) \cap A = \varnothing$

Theorem 18 Let (X, μ, I) be an ideal generalized topological space where (X, μ) is μ -locally indiscrete and μ is quasi-topology. If A, B are μ -separated and $I\mu$ -rg -open sets, then $A \cup B$ is $I\mu$ -rg -open.

Proof. Assume that A and B are μ -separated and $I\mu$ -rg -open sets, and let F be

μ -regular closed such that $F \subset A \cup B$. Since $F \cap c_\mu(A)$ and $F \cap c_\mu(B)$ are μ -closed and (X, μ) is μ -locally indiscrete, $F \cap c_\mu(A)$ and $F \cap c_\mu(B)$ are μ -regular closed. Since A and B is μ -separated, $F \cap c_\mu(A) \subset A$ and $F \cap c_\mu(B) \subset B$. By *Theorem 17,* $(F \cap c_\mu(A)) - U_1 \subset i_\mu(A)$ and $(F \cap c_\mu(B)) - U_2 \subset i_\mu(B)$ for some $U_1, U_2 \in I$. This implies $(F \cap c_\mu(A)) - i_\mu(A) \subset U_1$ and $(F \cap c_\mu(B)) - i_\mu(B) \subset U_2$. Then $(F \cap c_\mu(A)) - i_\mu(A) \in I$ and $(F \cap c_\mu(B)) - i_\mu(B) \in I$. Hence $((F \cap c_\mu(A)) - i_\mu(A)) \cup ((F \cap c_\mu(B)) - i_\mu(B)) \in I$, and so $(F \cap (c_\mu(A) \cup c_\mu(B)) - (i_\mu(A) \cup i_\mu(B))) \in I$. Since $F \subset A \cup B$, $F \subset c_\mu(F) \cap c_\mu(A \cup B)$ $= F \subset c_\mu(A \cup B)$, and so $F \subset F \cap c_\mu(A \cup B)$.

Thus
$$F - (i_\mu(A \cup B)) \subset F \cap c_\mu(A \cup B) - (i_\mu(A \cup B))$$
$$\subset F \cap (c_\mu(A) \cup c_\mu(B)) - (i_\mu(A) \cup i_\mu(B)).$$

Set $U = F \cap (c_\mu(A) \cup c_\mu(B)) - (i_\mu(A) \cup i_\mu(B))$. Then $U \in I$, and hence $F - (i_\mu(A \cup B)) \subset U$. So $F - U \subset (i_\mu(A \cup B))$, for some $U \in I$. By *Theorem 17,* $A \cup B$ is $I\mu$ -rg -open.

Corollary 19 Let (X, μ, I) be an ideal generalized topological space where (X, μ) is μ -locally indiscrete and μ is quasi-topology. Let A, B be $I\mu$ -rg -closed sets such that $X - A$ and $X - B$ are μ -separated in (X, μ). Then $A \cap B$ is $I\mu$ -rg -closed.

Proof. Let A and B be $I\mu$ -rg -closed set such that $X - A$ and $X - B$ are μ -separated in (X, μ). By *Theorem 18,* $(X - A) \cup (X - B)$ is $I\mu$ -rg -open. Then $X - (A \cap B)$ is $I\mu$ -rg -open, implies that $A \cap B$ is $I\mu$ -rg -closed.

Corollary 20 Let (X, μ, I) be an ideal generalized topological space where μ is a quasi-topology and let $A, B \subset X$. If A and B are $I\mu$ -rg -open sets in (X, μ), then $A \cap B$ is $I\mu$ -rg -open.

Proof. Assume that A and B are $I\mu$ -rg -open. Then $X - A$ and $X - B$ is $I\mu$ -rg -closed. By *Theorem 14*, $(X - A) \cup (X - B)$ is $I\mu$ -rg -closed. But $(X - A) \cup (X - B) = X - (A \cap B)$, we obtain that $X - (A \cap B)$ is $I\mu$ -rg -closed, which implies $A \cap B$ is $I\mu$ -rg -open.

Theorem 21 Let (X, μ, I) be an ideal generalized topological space and let A be an $I\mu$ -rg -open set. If $i_\mu(A) \subset B \subset A$, then B is $I\mu$ -rg -open in X.

Proof. Suppose that B is a subset of X such that $i_\mu(A) \subset B \subset A$. Since A is $I\mu$ -rg -open, $X - A \subset X - B \subset c_\mu(X - A)$ and $X - A$ is $I\mu$ -rg -closed. By *Theorem 9*, $X - B$ is $I\mu$ -rg -closed, and hence B is $I\mu$ -rg -open.

Theorem 22 Let (X, μ, I) be an ideal generalized topological space, where (X, μ) is μ-locally indiscrete. A subset A of X is $I\mu$-rg-closed if and only if $c_\mu(A) - A$ is $I\mu$-rg-open.

Proof. Assume that A is $I\mu$-rg-closed. Let F be μ-regular closed such that $F \subset c_\mu(A) - A$. By *Theorem 12*, $F \in I$. Set $U = F$. Then $U \in I$ and $F - U = \varnothing$. Clearly, $F - U \subset i_\mu(c_\mu(A) - A)$. By *Theorem 17*, $c_\mu(A) - A$ is $I\mu$-rg-open.

Conversely, assume that $c_\mu(A) - A$ is $I\mu$-rg-open. Let G be μ-regular open and $A \subset G$. Then $X - G \subset X - A$, and so $c_\mu(A) \cap (X - G) \subset c_\mu(A) \cap (X - A)$. Since $c_\mu(A) \cap (X - A) = c_\mu(A) - A$, we obtain that $c_\mu(A) \cap (X - G) \subset c_\mu(A) - A$. Since (X, μ) is μ-locally indiscrete, $c_\mu(A) \cap (X - G)$ is μ-regular closed. Since $c_\mu(A) - A$ is $I\mu$-rg-open, by *Theorem 17*, $(c_\mu(A) \cap (X - G)) - U \subset i_\mu(c_\mu(A) - A)$ for some $U \in I$. Since $A \subset G$ such that G is μ-regular open. Then A is μ- regular open. By *Theorem 3*, A is μ-closed, and hence $X - A$ is μ-open. Then $c_\mu(A) = A$. This shows that $c_\mu(A) \cap (X - A) = c_\mu(A) - A = \varnothing$. Then $i_\mu(c_\mu(A) - A) = \varnothing$, $(c_\mu(A) \cap (X - G)) - U \subset \varnothing$. Thus $c_\mu(A) \cap (X - G) \subset U \in I$. Therefore, by *Definition 6*, A is $I\mu$-rg-closed.

Discussion

This study introduces the concept of treasure generalized topological space (X, μ) with an ideal I on X. By which satisfy the following properties : (i) $A \in I$ and $B \subseteq A$ imply $B \in I$ and (ii) $A \in I$ and $B \in I$ imply $A \cup B \in I$. An ideal generalized topological space (X, μ) with an ideal I on X is denoted by (X, μ, I). Thus every μ-rg-closed set in (X, μ) is $I\mu$-rg-closed, but the converse need not be true. Then a generalized topological space (X, μ) is called μ-locally indiscrete if and only if every μ-closed set is μ-regular closed. Hence Then a subset A of X is $I\mu$-rg-closed if and only if $F \subset c_\mu(A) - A$ and F is μ-regular closed in X implies $F \in I$ and in general, the union of two $I\mu$-rg-closed set need not be a $I\mu$-rg-closed set. But μ is a quasi-topology then their union $A \cup B$ is also $I\mu$-rg-closed. Conversely, a subset A of X is said to be a μ-regular generalized open set with respect to I (briefly $I\mu$-rg-open set) if and only if $X - A$ is $I\mu$-rg-closed.

Conclusions

In this paper, we introduced the concept on a μ-regular generalized closed set with respect to an ideal (briefly $I\mu$-rg-closed set) We have found the properties μ-locally indiscrete and μ-separated in a

generalized topological space. We also studied $I\mu$ -rg -open sets in an ideal generalized topological space where generalized topological space is μ -locally indiscrete and quasi-topology.

Even though I have found several properties as presented in this paper, there are several questions yet to be answered and it may be worth investigating in future studies. I formulate the questions as follows:

1. Are there other properties of the ideal generalized topological space where generalized topological space is μ -locally indiscrete and quasi-topology?

2. Are there other properties of the separation axioms using the μ -regular generalized closed sets with respect to an ideal?

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