บททบทวนการออกแบบเชิงไลปูนอฟโดยใช้ฟังก์ชันควบคุมเชิงไลปูนอฟ
Review of Lyapunov Based Design via Control Lyapunov Functions

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บทคัดย่อ

ในบทความนี้ การสร้างฟังก์ชันควบคุมเชิงไลปูนอฟได้ถูกทบทวน วิธีการของฟังก์ชันควบคุมเชิงไลปูนอฟถูกอ้างถึงเพื่อแก้ปัญหาการควบคุมแบบเหมาะสมที่สุดชนิดที่เวลาไม่จำกัด เราอธิบายสมการเชิงอนุพันธ์ย่อยของแฮมิลตัน ยาโคบี เบลแมนและพบว่าคำตอบที่เหมาะสมที่สุดแบบย่อยของสมการนี้สามารถหาได้จากการใช้ฟังก์ชันควบคุมเชิงไลปูนอฟ ต่อมาการประยุกต์ของสูตรซองแทคเพื่อออกแบบตัวควบคุมแบบเหมาะสมที่สุดได้ถูกรวบรวมและสรุป บทความนี้อธิบายการสร้างฟังก์ชันควบคุมเชิงไลปูนอฟสำหรับระบบควบคุมที่ไม่เป็นเชิงเส้นซึ่งประกอบไปด้วย ระบบที่สามารถถูกทำให้เป็นเชิงเส้นแบบป้อนกลับและระบบที่มีอินทิเกรเตอร์แบคสเตปปิ้ง นอกจากนี้ตัวอย่างเพื่อประกอบคำอธิบายการออกแบบเชิงไลปูนอฟโดยใช้ฟังก์ชันควบคุมเชิงไลปูนอฟได้ถูกนำเสนอ

คำสำคัญ : ฟังก์ชันควบคุมเชิงไลปูนอฟ สูตรของซองแทค การทำให้เป็นเชิงเส้นแบบป้อนกลับ อินทิเกรเตอร์แบคสเตปปิ้ง

Abstract

In this paper, the construction of control Lyapunov functions (CLFs) for nonlinear systems is reviewed. A CLF approach is restated to solve the infinite-time optimal control problem. The Hamilton–Jacobi–Bellman (HJB) partial differential equation is illustrated and suboptimal solutions can be found by the use of CLFs. Further, the application of the generalization of Sontag's formula to design an optimal feedback stabilizing controller is briefly summarized. The construction of CLFs for several special classes of nonlinear systems including feedback linearization and integrator backstepping is explained with simplified expressions of developed theories. Examples are also presented to illustrate Lyapunov-based controller design techniques using a CLF.

Keywords : Control Lyapunov function (CLF), Sontag's formula, feedback linearization, integrator backstepping

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Introduction

Lyapunov functions are used to show a sufficient condition for proving the stability of a dynamical system (for a reference see (Khalil, 1992 ; Hahn, 1967 ; Rouche *et al*., 1977)). This energy-like function must always be a positive-definite function of the state, and it must be decreasing along trajectories of the state. Unfortunately, the Lyapunov functions prove stability for closed-loop systems, where the control law is already known. To harness the power of this theory for the purpose of control synthesis, Lyapunov's second method was extended to dynamical systems with inputs by Artstein (Artstein, 1983) and Sontag (Sontag, 1983) with the introduction of control Lyapunov functions (CLFs). Similar to Lyapunov stability theorem the existence of a CFL is also a necessary and sufficient condition for the stabilizability of nonlinear systems with control inputs. For nonlinear optimal control problem, it has shown that a standard dynamic programming approach reduces the problem to the HJB partial differential equation. In other word, to solve optimal control problems is equivalent to solve the HJB equation. However, it is very complicate to solve the HJB equation for nonlinear dynamic systems. Hence suboptimal solutions obtained by using the CLF concept are considered. In contrast with traditional Lyapunov functions, a CLF can be defined for a system with inputs without specifying a particular feedback function. Sontag (Sontag, 1989) has shown that, if a CLF is known for a nonlinear system, then the CLF and the system equations can be used to find a controller that makes the system asymptotically stable. Freeman and Kokotovi (Freeman & Kokotovi, 1996) have shown that every CLF solves the HJB equation associated with a meaningful cost. In other words, if a CLF exists for a nonlinear system, we can compute the resulting optimal control law without solving the HJB equation.

Finding a CLF for a general nonlinear system is an open problem. For several special classes of nonlinear systems, CLFs can be founded. Feedback linearization (Lin & Sontag,1991; Malisoff & Sontag, 1997) can be used to construct a CLF when the system dynamics can be transformed into a linear structure. Likewise, integrator backstepping (Sepulchre *et al*., 1997; Krstic *et al*., 1975; Krstic & Li, 1998) will generate a CLF whenever the system can be put into a cascade structure. In this paper we summarize the application of a CLF to design an optimal feedback stabilizing controller and review the construction of CLFs for special classes of nonlinear systems. Furthermore, we give discussions of subsequent researches involving CLFs that may be performed in the future.

Problem Formation

Consider the following optimal control problem

$$\min_{u(\cdot)} \int_0^\infty [q(x)+u^2]dt \tag{1}$$

$$s.t.\ \frac{dx}{dt} = f(x)+g(x)u \tag{2}$$

where $x(t)\in\Re^n$ denotes the state, $u(t)\in\Re$ represents the control and $f(x)\in\Re^n$ is a sufficiently smooth function of the state vector $x(t)$, and $x(0)$ is the initial condition of the process. $q(x)$ is continuously differentiable, symmetric positive definite and (f, q) is zero-detectable.

The aim is to determine the control signal u to solve the system (2) and minimize the performance index (1). Next the procedure to derive the HJB equation presented by Primbs and his co-researchers (Primbs *et al*., 1999) is restated.

Using a standard dynamic programming argument, the HJB equation for the above problem can be written as

$$V_x^* f - \frac{1}{4}V_x^* g g^T V_x^{*T} + q(x) = 0 \tag{3}$$

where $V_x^* = \left[\frac{\partial V^*}{\partial x_1}, \ldots, \frac{\partial V^*}{\partial x_n}\right]$. V^* is the minimum cost to go from the current state $x(t)$, *i.e.*,

$$V^*(x(t)) = \inf_{u(\cdot)} \int_t^{\infty} [q(x(\tau)) + u^2(\tau)]d\tau \quad (4)$$

If a continuously differentiable, positive solution to the HJB equation (3) exists, then the optimal control input is given by

$$u^* = -\frac{1}{2}V_x^* g(x) \quad (5)$$

At this stage, the HJB equation (3) solves the optimal control problem for every initial condition all at once. Hence, it is a global approach in this sense and offers a closed-loop feedback formula for the optimal controller. However, the HJB equation is extremely difficult to solve analytically. We alternatively seek a suboptimal solution. Thus, the basic concepts of a CLF (Primbs *et al*., 1999) to obtain a suboptimal solution are also given below.

Definition 1 (Primbs et al., 1999): A continuously differentiable positive definite function $V(x)$ is called a Control Lyapunov Function (CLF) for system (2) if for $x \in \Re^N$ and $x \neq 0$, $V_x g = 0 \Rightarrow V_x f < 0$.

We assume that $V(x)$ is a CLF for the system (2) and $V(x)$ possesses the same shape level curves as those of the value function V^*. This implies a relationship between the gradients of V^* and V. In such a circumstance, there exists a scalar function $\lambda(x)$ such that $V^*_x = \lambda(x)V_x$ for every x. Thus the optimal controller (5) can also be rewritten as

$$u^* = -\frac{1}{2}V_x^* g(x) = -\frac{\lambda(x)}{2}V_x g(x). \quad (6)$$

In addition, substituting $V^*_x = \lambda(x)V_x$ into the HJB equation (3), $\lambda(x)$ can be determined by

$$\lambda(x)V_x f - \frac{1}{4}(\lambda(x))^2 V_x g g^T V_x^T + q(x) = 0. \quad (7)$$

Solving (7) and taking only the positive square root, yields

$$\lambda(x) = 2\left(\frac{V_x f + \sqrt{(V_x f)^2 + q(x)[V_x g g^T V_x^T]}}{V_x g g^T V_x^T}\right). \quad (8)$$

Substituting (8) into (6), then the controller u^* becomes

$$u^* = \begin{cases} -\left(\dfrac{V_x f + \sqrt{(V_x f)^2 + q(x)(V_x g g^T V_x^T)}}{V_x g g^T V_x^T}\right) g^T V_x^T, & V_x g \neq 0 \\ 0, & V_x g = 0 \end{cases} \quad (9)$$

which is known as Sontag's formula (Sontag, 1989). Note that u^* is bounded when $V_x g$ goes to zero. Under this control input it can be found that

$$\dot{V} = V_x(f(x) + g(x)u) \quad (10)$$

Substituting u^* into (10), we obtain

$$\dot{V} = -\sqrt{(V_x f)^2 + q(x)(V_x g g^T V_x^T)}. \quad (11)$$

Obviously, $\dot{V} < 0$ is ensured and one can conclude that this controller yields global asymptotic stability.

Construction of CLFs for a linear system

In this section we review a method to construct a CLF for a linear system

$$\dot{x} = Ax + Bu \quad (12)$$

where $x \in \Re^n$ denotes the state, $u \in \Re$ represents the control, $A \in \Re^{n\times n}$ is a constant matrix and $B \in \Re^n$ denotes a constant vector.

Freeman and Primbs (Freeman and Primbs, 1996) showed that a CLF for the system (12) is

$$V = x^T P x \quad (13)$$

where $P = P^T > 0$ is the unique solution of the Riccati equation

$$PA + A^T P + Q - PBB^T P = 0 \quad (14)$$

with any given $Q = Q^T > 0$ and $q(x) = x^T Qx$. It implies the feedback

$$u = -B^T P x \quad (15)$$

By the feedback above the closed-loop system becomes

$$\dot{x} = (A - BB^T P)x \quad (16)$$

With (13), the time derivative of V is

$$\dot{V} = x^T(A^T P + PA - 2PBB^T P)x \quad (17)$$

We now show that the quadratic Lyapunov function (13) is a CLF for the system (12). With the condition $V_x g = V_x B = 0$, one obtains

$$B^T Px = 0, \qquad x \neq 0 \tag{18}$$

Substituting (18) into (17), we obtain

$$x^T (A^T P + PA)x = V_x f < 0 \tag{19}$$

Therefore, we obtain $V_x f < 0$ whenever $V_x g = 0$. Thus, the quadratic Lyapunov function (13) is a CLF for the system (12). After obtaining a CLF for the system (12) the Sontag's formula can be used to design a stabilizing control law

$$u^* = \begin{cases} -\left(\dfrac{x^T PAx + \sqrt{(x^T PAx)^2 + 4(x^T PB)^4}}{x^T PB} \right), & x^T PB \neq 0 \\ 0 & , \ x^T PB = 0. \end{cases} \tag{20}$$

For linear systems, the unique solution P results in the optimal value function V. However, in general, this relation does not hold for nonlinear systems.

Construction of CLFs for feedback linearizable systems

In this section, we summarize a method to find a CLF for feedback linearizable systems by using of the linear transformation technique. Feedback linearization is a significant method to nonlinear control design. The main idea of this scheme is to find a state transformation $z = \Phi(x)$ and an input transformation $u = u(x, v)$ so that the nonlinear system dynamics is transformed into equivalent linear time-invariant systems. After obtaining the form $\dot{z} = Az + Bv$, then linear control technique can be applied.

The most convenient structure for a static state feedback control is of the form

$$u = \alpha(x) + \beta(x)v \tag{21}$$

where v is the external reference input. In fact the composition of this control with a system of a form

$$\dot{x} = f(x) + g(x)u \tag{22}$$
$$y = h(x)$$

yields a closed-loop characterized by the similar structure

$$\dot{x} = f(x) + g(x)\alpha(x) + g(x)\beta(x)v \tag{23}$$
$$y = h(x).$$

Consider a nonlinear system having relative degree $r = n$, i.e. exactly equal to the dimension of the state space. The change of coordinates required to construct to normal form is given exactly by

$$\Phi(x) = \begin{bmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \cdots \\ \varphi_n(x) \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h(x) \\ \cdots \\ L_f^{n-1} h(x) \end{bmatrix}. \tag{24}$$

Note that the Lie derivative of $h(x)$ with respect to $f(x)$ is denoted by $L_f h = \left[\dfrac{\partial h(x)}{\partial x}\right]^T f(x)$. The following notations are often used to simplify expressions involving iterated Lie derivatives,

$$L_f^2 h = \left[\frac{\partial(L_f h(x))}{\partial x}\right]^T f(x) \text{ and } L_f^k h = \left[\frac{\partial(L_f^{k-1} h(x))}{\partial x}\right]^T f(x).$$

No extra functions are needed to complete the transformation. We obtain new coordinates as

$$z_i = \phi_i(x) = L_f^{i-1} h(x), \quad i = 1, \ldots, n \tag{25}$$

Letting $z = [z_1, z_2, \ldots, z_n]^T$, the system is now described by equation

$$\dot{z} = Az + Bv, \tag{26}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{27}$$

and

$$y = h(\Phi^{-1}(z)). \tag{28}$$

We refer to this as input-state linearization. We are also interested in feedback laws that linearize input/output maps while leaving the state equations partially linearized. When a system is transformed to the form (26), the method in Section 3, can be used to determine a CLF.

Construction of CLFs by using the integrator backstepping technique

In this section a CLF for a more general class of nonlinear systems with the integrator backstepping technique is reviewed. Consider the system in control affine form with an integrator at the input

$$\dot{x} = F(x) + G(x)\xi \qquad \dot{\xi} = u \tag{29}$$

where $x \in \Re^n, \xi \in \Re$ are the states, and $u \in \Re$ is the control input. We want to design a feedback controller to stabilize the origin $(x = 0, \xi = 0)$.

The system can be seen as the cascaded connection of two components. Suppose that the first component can be stabilized by a feedback law $\xi = \alpha(x)$ with $\alpha(0) = 0$, so that the origin of

$$\dot{x} = F(x) + G(x)\alpha(x) \tag{30}$$

is asymptotically stable.

Suppose that we know a CLF $V(x)$ that satisfies

$$\dot{V} = \frac{\partial V}{\partial x}\left[F(x) + G(x)\alpha(x)\right] \le -W(x) \tag{31}$$

where $W(x)$ is a positive definite function. Adding and subtracting $G(x)\alpha(x)$ on the right hand side of (30), we obtain

$$\begin{aligned} \dot{x} &= \left[F(x) + G(x)\alpha(x)\right] + G(x)\left[\xi - \alpha(x)\right] \\ \dot{\xi} &= u. \end{aligned} \tag{32}$$

Letting $\omega = \xi - \alpha(x)$ and substituting this into (32), we obtain

$$\begin{aligned} \dot{x} &= \left[F(x) + G(x)\alpha(x)\right] + G(x)\omega \\ \dot{\omega} &= u - \dot{\alpha}. \end{aligned} \tag{33}$$

Note that ω represents the difference between the input ξ and the desired input $\alpha(x)$. The derivative of α can be computed using

$$\dot{\alpha} = \frac{\partial \alpha}{\partial x}\left[F(x) + G(x)\xi\right]. \tag{34}$$

Taking $v = u - \dot{\alpha}$ reduces the system to

$$\begin{aligned} \dot{x} &= \left[F(x) + G(x)\alpha(x)\right] + G(x)\omega \\ \dot{\omega} &= v \end{aligned} \tag{35}$$

which is similar to the system from which we began, except that now the first component is asymptotically stable when the input is zero.

A Lyapunov function candidate can be chosen as

$$V_c(x) = V(x) + \frac{1}{2}\omega^2. \tag{36}$$

The first time derivative of V_c is

$$\begin{aligned} \dot{V}_c &= \frac{\partial V}{\partial x}\left[F(x) + G(x)\alpha(x)\right] + \frac{\partial V}{\partial x}G(x)\omega + \omega v \\ &\le -W(x) + \frac{\partial V}{\partial x}G(x)\omega + \omega v. \end{aligned} \tag{37}$$

Selecting $v = -\frac{\partial V}{\partial x}G(x) - k\omega,\ k > 0$ and substituting this into (37), we obtain

$$\dot{V}_c \le -W(x) - k\omega^2. \tag{38}$$

Clearly, $\dot{V}_c$ is negative definite, so the origin $x = 0$ is asymptotically stable. Using $\alpha(0) = 0$ we can conclude that the origin $x = 0$, $\xi = 0$ is also asymptotically stable. This leads to the conclusion of a CLF. Hence, it can be seen that the Lyapunov function V_c defined in (36) is a CLF for the system (29).

Examples

We present examples which illustrate constructions of CLFs for a feedback linearizable system and a system with an integrator at the input.

Example I

Consider

$$\begin{aligned} \dot{x}_1 &= a\sin(x_2) \\ \dot{x}_2 &= -x_1^2 + u \\ y &= x_2 \end{aligned} \tag{39}$$

where a is a constant. Suppose that we want to find a CLF for the system above. We begin by finding

$$h(x) = y = x_2$$

and

$$z_1 = x_1$$

$$z_2 = L_f h(x) = \frac{\partial h(x)}{\partial x} f(x) = [1\ \ 0]\begin{bmatrix} a\sin(x_2) \\ -x_1^2 \end{bmatrix} = a\sin(x_2) = \dot{x}_1 \tag{40}$$

This generates the transformed state system

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = a\cos(x_2)(-x_1^2 + u) \quad (41)$$

and the nonlinearity may now be cancelled by the control

$$u = x_1^2 + \frac{1}{a\cos(x_2)}v. \quad (42)$$

Now this particular transformation is invertible for $-\frac{\pi}{2} < x_2 < \frac{\pi}{2}$ and we can express x_1 and x_2 in terms of z_1 and z_2 as follows,

$$x_1 = z_1$$
$$x_2 = \sin^{-1}(\frac{z_2}{a}). \quad (43)$$

Substituting (43) into (41) yields

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = a\cos(\sin^{-1}(\frac{z_2}{a}))(-z_1^2 + u) \quad (44)$$

where

$$u = x_1^2 + \frac{1}{a\cos(x_2)}v$$
$$= z_1^2 + \frac{1}{a\cos(\sin^{-1}(\frac{z_2}{a}))}v \quad (45)$$

Inserting (45) into the transformed system yields,

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = v, \quad (46)$$

which is completely linearized. To find a CLF, the system is now described by equation

$$\dot{z} = Az + Bv \quad (47)$$

where

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (48)$$

Therefore, using (13) a CLF for the above system is

$$V = z^T P z \quad (49)$$

where $P = P^T > 0$ is the unique solution of the Riccati equation

$$PA + A^T P + Q - PBB^T P = 0. \quad (50)$$

Note that when we obtain a CLF, the Sontag's formula can be used to design a stabilizing control law.

Example II

Consider the system

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2$$
$$\dot{x}_2 = u \quad (51)$$

and now suppose that in the first component x_1 is viewed as an input. A feedback controller $x_2 = \alpha(x_1)$ is designed to stabilize $x_1 = 0$.

With the following feedback

$$x_2 = \alpha(x_1) = -x_1^2 - x_1, \quad (52)$$

we cancel the nonlinear term x_1^2 to obtain

$$\dot{x}_1 = -x_1 - x_1^3 \quad (53)$$

and $V(x_1) = \frac{1}{2}x_1^2$ satisfies

$$\dot{V} = -x_1^2 - x_1^4 \le -x_1^2, \quad (54)$$

so that $x_1 = 0$ is globally exponentially stable.

To apply the integrator backstepping technique, we use the change of variables

$$z_2 = x_2 - \alpha(x_1) = x_2 + x_1 + x_1^2 \quad (55)$$

to transform the system into the form

$$\dot{x}_1 = -x_1 - x_1^3 + z_2$$
$$\dot{z}_2 = u + (1 + 2x_1)(-x_1 - x_1^3 + z_2) \quad (56)$$

Consider the augmented control Lyapunov function

$$V_c(x) = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2. \quad (57)$$

Differentiating V_c gives

$$\dot{V}_c = -x_1^2 - x_1^4 + z_2(x_1 + (1 + 2x_1)(-x_1 - x_1^3 + z_2) + u) \quad (58)$$

Taking $u = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z_2) - z_2$ gives

$$\dot{V}_c = -x_1^2 - x_1^4 - z_2^2 \quad (59)$$

Hence the origin is globally asymptotically stable and we obtain V_c defined in (57) as a CLF for the system (51).

Discussions on the main limitation and future researches of the CLF approach

The difficulty of the CLF scheme is to find a CLF because theory developed to find a CLF for a general nonlinear system has not appeared, but feedback linearization can be used to construct a CLF when the system dynamics can be transformed into a linear structure. Similarly, the integrator backstepping technique can be applied to generate a CLF whenever the system can be put into a cascade structure. However, practical system designs of general applications are normally involved with various classes of nonlinear systems. This is the reason why the CLF approach is not popular to be used in real-life applications. We believe that the future studies regarding the CLF approach will focus on the theory development to find a CLF for other classes of nonlinear systems. Once a CLF can be found for a general class of nonlinear systems, researches on practical applications of this method to design feedback stabilizing control laws will be later conducted.

Conclusion

In this paper reviews of synthesizing state feedback controller using the CLF method and the construction of CLFs for some special classes of nonlinear systems have been proposed. For practical implementation it is always difficult to find a CLF specifically for each nonlinear system. Due to the limitation of this method, it is rarely applied to design suboptimal controllers for practical nonlinear systems. Examples are presented to demonstrate the construction of CLFs for a feedback linearizable system and a system with integrator backstepping.

References

Artstein, Z. (1983). Stabilization with relaxed controls. *Nonlinear Analysis, Theory, Methods, and Applications, 7*(11), 1163-1173.

Freeman, R.A. & Kokotovic, P.V. (1996). Inverse optimality in robust stablilzation. *SIAM Journal of Control and Optimization, 34*(4), 1365-1391.

Freeman, R.A. & Primbs, J.A. (1996). Control Lyapunov function: new ideas from an old source. In *Proceedings of the 35th IEEE Decision and Control Conference*. (pp. 3926-3931). Kobe, Japan,

Hahn, W., (1967). *Stability of Motion*. New York: Springer-Verlag.

Khalil, H. K. (1992). *Nonlinear Systems*. New York: Macmillan Publishing Company.

Krstic, M., Kanellakopoulos, I. & Kokotovic, P. V. (1995). *Nonlinear and Adaptive Control Design*. New York : John Wiley& Sons.

Krstic, M. & Li, Z.-H. (1998). Inverse optimal design of input-to-state stabilizing nonlinear controllers. *IEEE Transactions on Automatic Control, 43*(3), 336-350.

Lin, Y. & Sontag, E. D. (1991). A universal formula for stabilization with bounded controls, *Systems Control Letters, 16*, 393-397.

Malisoff, M. & Sontag, E. D. (1997). Universal formulas for CLF's with respect to Minkowski balls, In *Proceedings of the European Control Conference*. Brussels, Belgium.

Primb, J. A., Nevistic, V. & Doyle, J. C. (1999). Nonlinear optimal control: A control Lyapunov function and receding horizon perspective. *Asian Journal of Control, 1*, 14-24.

Rouche, N., Habets, P. & Laloy, M. (1977). *Stability Theory by Lyapunov's Direct Method*: Springer-Verlag.

Sepulchre R., Jankovic, M. & Kokotovic, P. V. (1997). *Constructive Nonlinear Control*. Communication and Control Engineering Series. Berlin-Heidelberg-New York: Springer-Verlag.

Sontag, E. D. (1983). A Lyapunov-like characterization of asymptotic controllability. *SIAM Journal on Control and Optimization, 21*, 462-471.

Sontag, E. D. (1989). A universal construction of Artstein's theorem on nonlinear stabilization. *Systems and Control Letters, 13*, 117-123.